Ramsey Theory on Infinite Structures

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Outline: Day 1

I. Ramsey Theory on Sets

II. Ramsey Theory on the Rationals

III. Structural Ramsey Theory and Big Ramsey Degrees

IV. First 3 Ingredients of a Big Ramsey Degree

   (a) Enumerated structures and their coding trees of 1-types
   (b) Diagonal Antichains
   (c) Passing Numbers

V. The Halpern-Läuchli Theorem and Harrington's 'forcing proof'

Outline: Day 2

I. Big Ramsey Degree Characterizations and Methods

   (a) $\mathbb{Q}$, Rado, unrestricted: Milliken's Strong Tree Theorem
   (b) SDAP$^+$ and FAP Structures: Forcing on Coding Trees
   (c) Generic Poset: Parameter Words

II. Infinite-dimensional Ramsey Theory on ω

III. Infinite-dimensional Structural Ramsey Theory

IV. More Directions

I. Ramsey Theory on Sets

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Theorem (Finite Pigeonhole Principle)

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Figure: 10 pigeons in 9 holes

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Theorem (Infinite Pigeonhole Principle)

If infinitely many marbles are placed into finitely many buckets, then some bucket contains infinitely many marbles.

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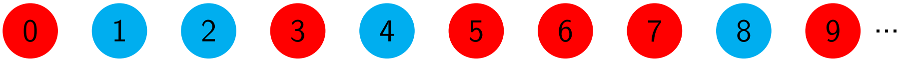
Theorem (Infinite Pigeonhole Principle)

Given a coloring of the natural numbers into finitely many colors, at least one color class is infinite.



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For $m < n$ and $2 \leq r$, there is a p large enough so that for any coloring $\chi : [p]^m \to r$, there is an $N \subseteq [p]^n$ such that χ takes one color on $[N]^m$.

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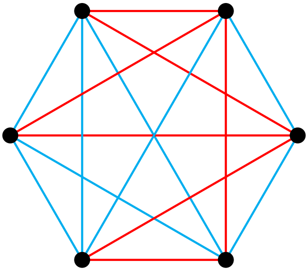

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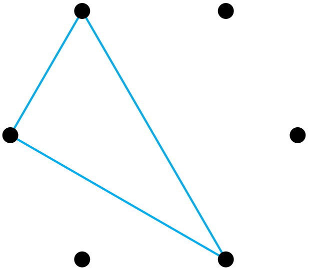

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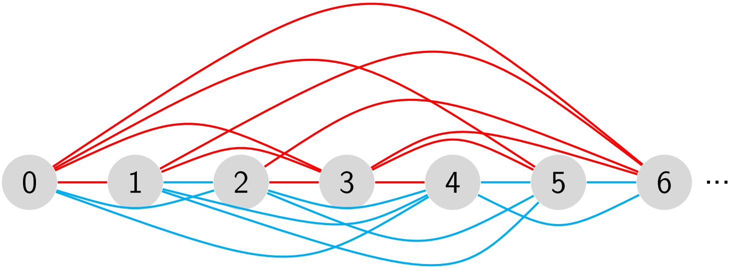

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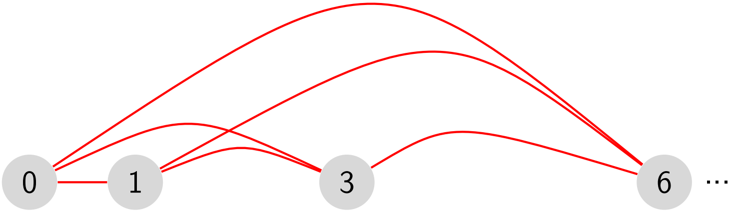

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Structural interpretations:

- coloring hyper edges in a complete m-regular hypergraph on infinitely many vertices
- coloring linear orders of size m inside $(\mathbb{N}, <)$

Which infinite structures carry analogues of Ramsey's Theorem?

II. Ramsey Theory on the Rationals

The Rationals as a Dense Linear Order

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$-3/2$	-1	$-1/2$	0	$1/2$	$3/4$	1	$3/2$
q_2	q_7	q_1	q_0	q_3	q_6	q_5	q_4

These patterns are unavoidable.

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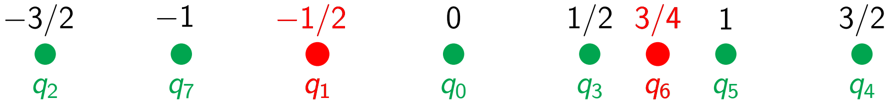

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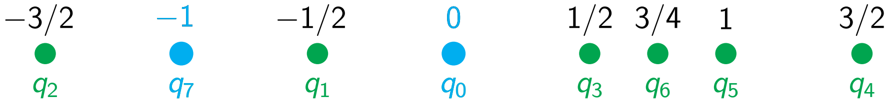

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Coloring Finite Sets of Rationals

Theorem (D. Devlin, 1979)

Given m, if $[\mathbb{Q}]^m$ is colored by finitely many colors, then there is a subcopy $\mathbb{Q}' \subseteq \mathbb{Q}$ forming a dense linear order such that $[\mathbb{Q}']^m$ take no more than $C_{2m-1}(2m-1)!$ colors. This bound is optimal.

m	**Bound**
1	1
2	2
3	16
4	272

C_i is from

$$\tan(x) = \sum_{i=0}^{\infty} C_i x^i$$

- Galvin (1968) The bound for pairs is two.
- Laver (1969) Upper bounds for all finite sets.

Which other infinite structures carry analogues of Ramsey's Theorem?

Background: Finite Structural Ramsey Theory

For structures $\mathbf{A}, \mathbf{B}$, write $\mathbf{A} \le \mathbf{B}$ iff $\mathbf{A}$ embeds into $\mathbf{B}$.

$\binom{\mathbf{B}}{\mathbf{A}}$ denotes the set of all copies of $\mathbf{A}$ in $\mathbf{B}$.

A class $\mathcal{K}$ of finite structures has the **Ramsey Property** if given $\mathbf{A} \le \mathbf{B}$ in $\mathcal{K}$ and r, there is $\mathbf{C} \in \mathcal{K}$ so that

$$\forall \chi : \binom{\mathbf{C}}{\mathbf{A}} \to r \quad \exists \mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}},\ \chi \restriction \binom{\mathbf{B}'}{\mathbf{A}} \text{ is constant.}$$

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Lots of work done! (e.g., Nešetřil–Rödl, Hubička–Nešetřil)

Examples: The classes of **finite** linear orders, ordered graphs, ordered k-clique-free graphs, ordered k-regular hypergraphs, partial orders with linear extension,...

Passing Remark.

Take the orders away and you get small Ramsey degrees.

III. Structural Ramsey Theory and Big Ramsey Degrees

Universal and homogeneous structures.

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- Fraïssé correspondence between $\mathcal{K}$ and its limit $\mathbf{K}$ being homogeneous and universal for $\mathcal{K}$.
- Any two homogeneous structures which are universal for $\mathcal{K}$ are isomorphic.

Let **K** be an infinite structure.

K has **finite big Ramsey degrees** if for each finite $\mathbf{A} \le \mathbf{K}$, $\exists T$ such that $\forall r$, $\forall\, \chi : \binom{\mathbf{K}}{\mathbf{A}} \to r$, $\exists \mathbf{K}' \in \binom{\mathbf{K}}{\mathbf{K}}$ such that $|\chi \restriction \binom{\mathbf{K}'}{\mathbf{A}}| \le T$.

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The **big Ramsey degree** of **A** in **K**, $T(\mathbf{A})$, is the least such T.

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- $\mathbf{K}$ has the exact analogue of Ramsey's Theorem iff $T(\mathbf{A}) = 1$ for all $\mathbf{A} \in \mathcal{K}$.
- Except for vertex colorings, this usually fails: If $|\mathrm{Aut}(\mathbf{K})| > 1$, then $\exists \mathbf{A} \in \mathcal{K}$ with $T(\mathbf{A}) > 1$, or infinite. (Hjorth 2008)

Topological Dynamics and Ramsey Theory

Theorem (Kechris–Pestov–Todorcevic, 2005)

A Fraïssé class $\mathcal{K}$ of finite structures has the Ramsey property if and only if $Aut(\mathbf{K})$ is extremely amenable, where $\mathbf{K}$ is the homogeneous structure universal for $\mathcal{K}$.

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Theorem (Zucker, 2019)

If $\mathbf{K}$ has a big Ramsey structure, then $Aut(\mathbf{K})$ admits a unique universal completion flow.

Big Ramsey Degree results, a sampling

- 1933. $T(\text{Pairs}, \mathbb{Q}) \geq 2$. (Sierpiński)
- 1975. $T(\text{Edge}, \mathcal{R}) \geq 2$. (Erdős, Hajnal, Pósa)
- 1979. $(\mathbb{Q}, <)$: All BRD computed. (D. Devlin)
- 1986. $T(\text{Vertex}, \mathcal{H}_3) = 1$. (Komjáth, Rödl)
- 1989. $T(\text{Vertex}, \mathcal{H}_n) = 1$. (El-Zahar, Sauer)
- 1996. $T(\text{Edge}, \mathcal{R}) = 2$. (Pouzet, Sauer)
- 1998. $T(\text{Edge}, \mathcal{H}_3) = 2$. (Sauer)
- 2006, 2008. The Rado graph: All BRD characterized; computed. (Laflamme, Sauer, Vuksanović); (J. Larson)
- 2010. Dense Local Order $\mathbf{S}(2)$ and $\mathbb{Q}_n$: All BRD computed. (Laflamme, Nguyen Van Thé, Sauer)

Developments via coding trees and forcing (arxiv dates)

- 2017. Triangle-free Henson graphs: Very good Bounds. Exact bounds via small tweak in 2020. (D.)
- 2019. k-clique-free Henson graphs: Upper Bounds. (D.)
- 2020. Finitely constrained binary FAP: Upper Bounds. (Zucker)
- 2020. Exact BRD for binary (Part I) and indivisibility for higher arity (Part II) SDAP^+ structures. (Coulson, D., Patel)
- 2021. Binary rel. Forb($\mathcal{F}$): Exact BRD. (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker)
- Also some ∞-dimensional Ramsey theorems (tomorrow).

Why forcing?

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Why coding trees? (soon)

Developments not using forcing (arxiv dates)

- 2018. Certain homogeneous metric spaces: Upper Bounds. (Mašulović) category theory.
- 2019. 3-uniform hypergraphs: Upper Bounds. (Balko, Chodounský, Hubička, Konečný, Vena) Milliken Theorem.
- 2020. Circular directed graphs: Exact BRD Computed. (Dasilva Barbosa) category theory.
- 2020. Homogeneous partial order: Upper Bounds. (Hubička) Ramsey space of parameter words. First non-forcing proof for $\mathcal{H}_3$.
- 2021. Homogenous graphs with forbidden cycles (metric spaces): Upper Bounds. (Balko, Chodounský, Hubička, Konečný, Nešetřil, Vena) parameter words.
- 2023. Homogeneous partial order: Exact BRD. (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker) parameter words.
- 2023+. Certain Forb($\mathcal{F}$) binary and higher arities. (BCDHKNVZ) New methods.
- And more...

What is a big Ramsey degree?

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We give a current (non-historical) perspective today.

IV. First 3 Ingredients of a Big Ramsey Degree

First Ingredient: Enumerating the Universe

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All substructures of **K** have some memory of this enumeration.

Enumerating the universe of $\mathbf{K}$ in order-type ω induces a coding tree of 1-types.

All substructures of $\mathbf{K}$ have some memory of this enumeration.

Let $\mathbf{K}$ be a homogeneous structure with vertices $\langle v_i : i < \omega \rangle$.

Let $\mathbf{K}_n = \mathbf{K} \restriction \{v_i : i < n\}$.

This **coding tree of 1-types** $\mathbb{S}(\mathbf{K})$ is the set of all (quantifier free) complete 1-types over $\mathbf{K}_n$, $n < \omega$, along with a function $c : \omega \to \mathbb{S}(\mathbf{K})$ where $c(n)$ is the 1-type of v_n over $\mathbf{K}_n$. The tree-ordering is inclusion.

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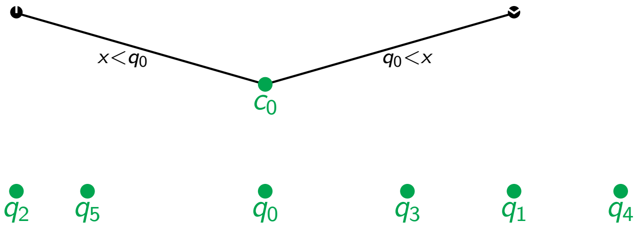

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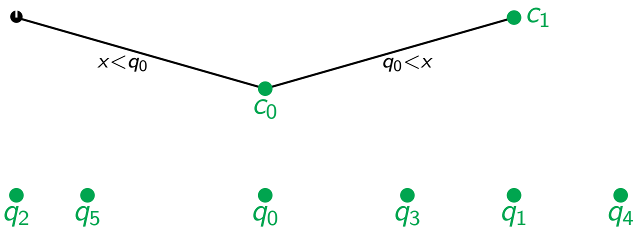

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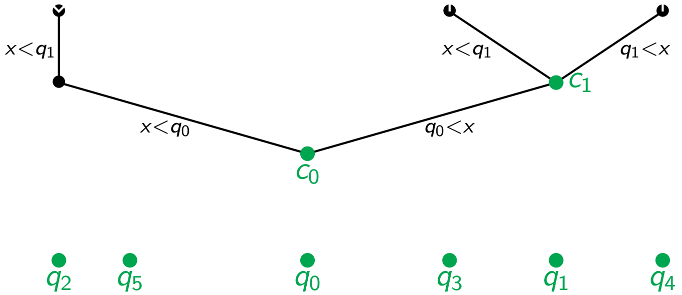

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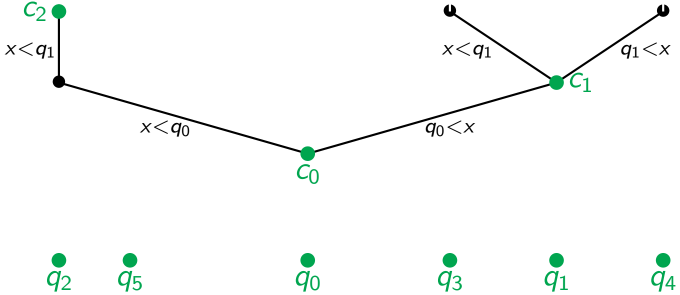

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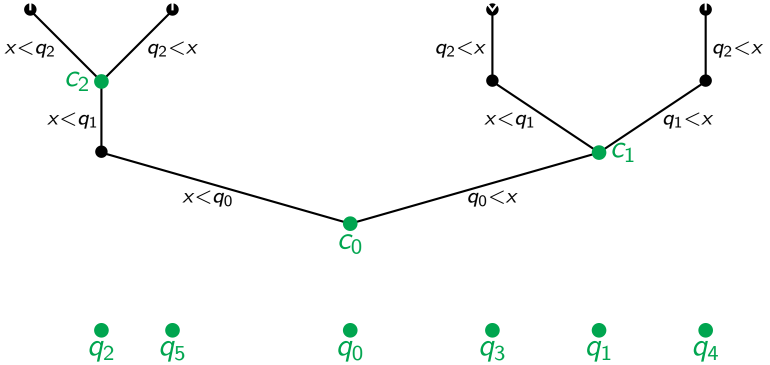

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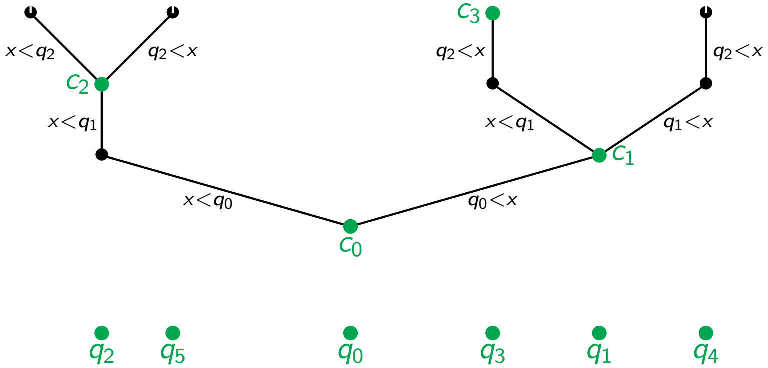

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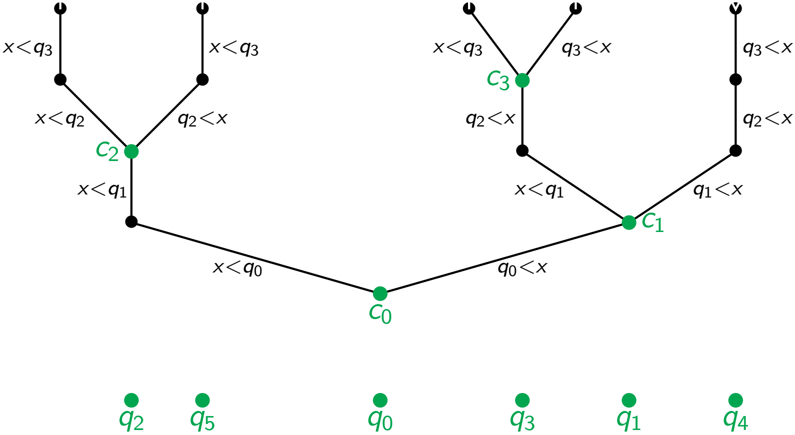

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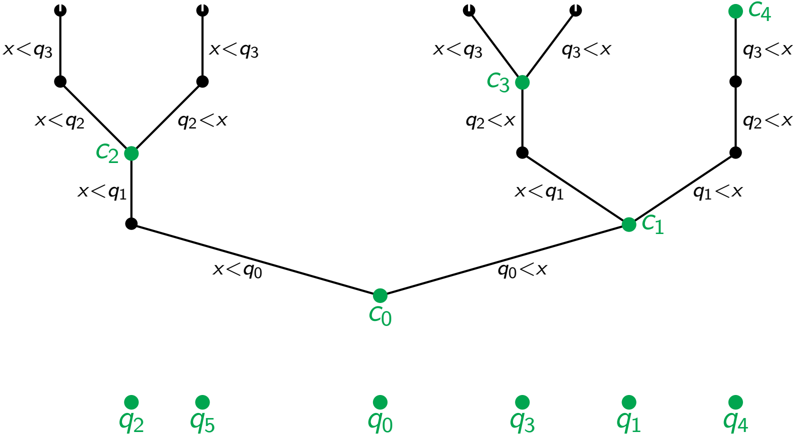

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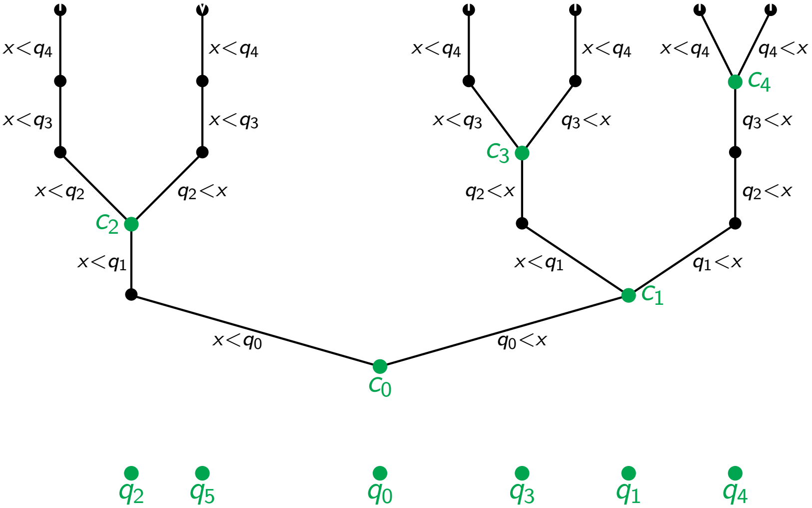

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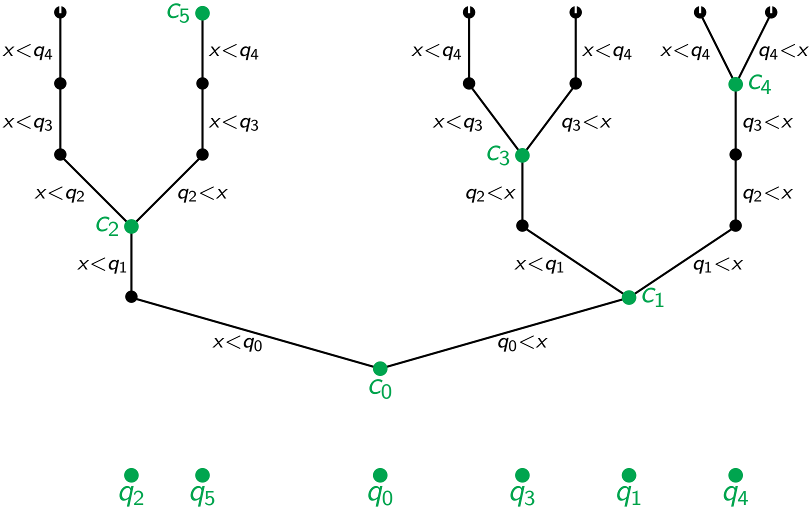

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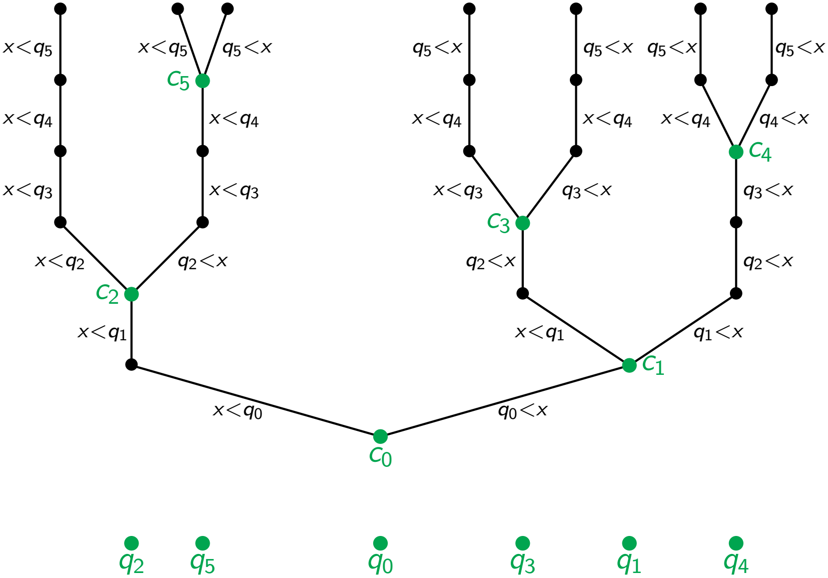
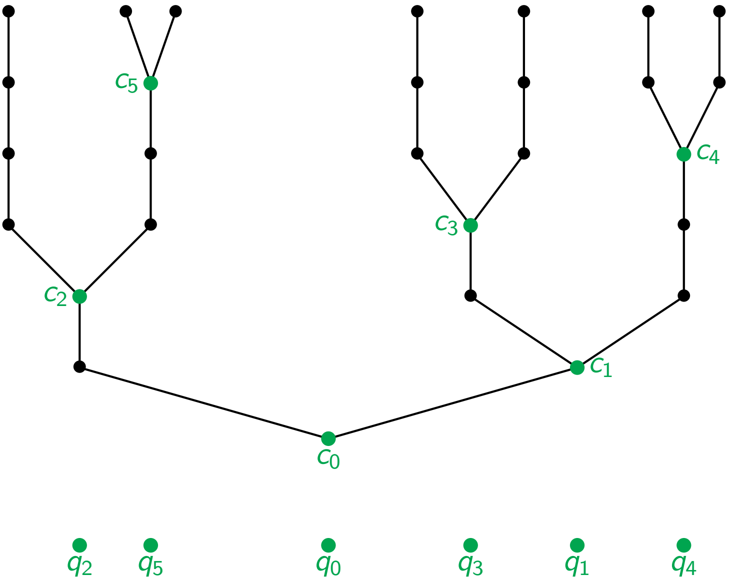


Sierpiński's Coloring



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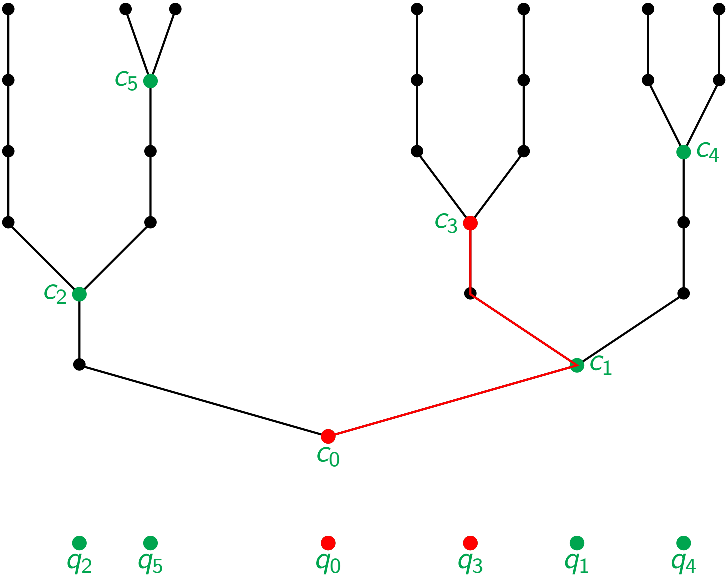

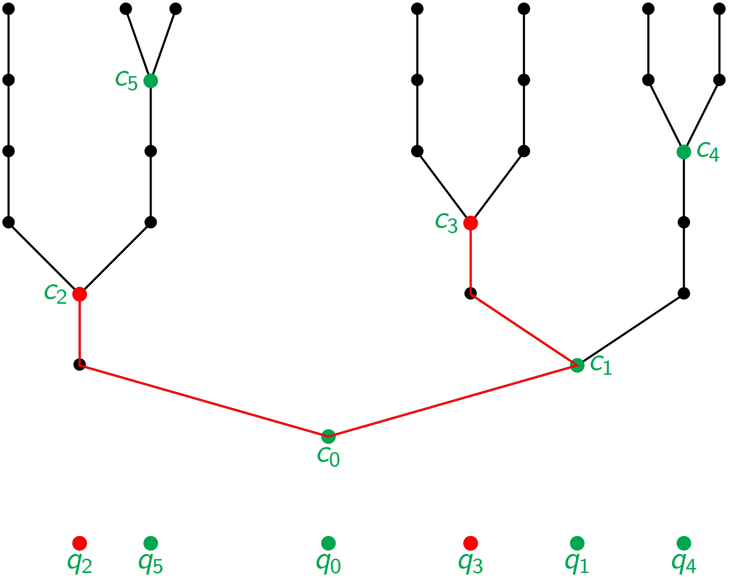
c_5
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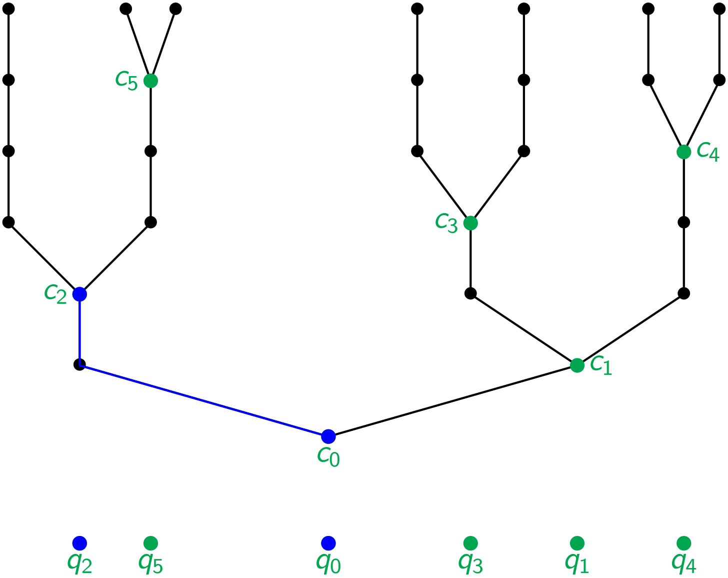

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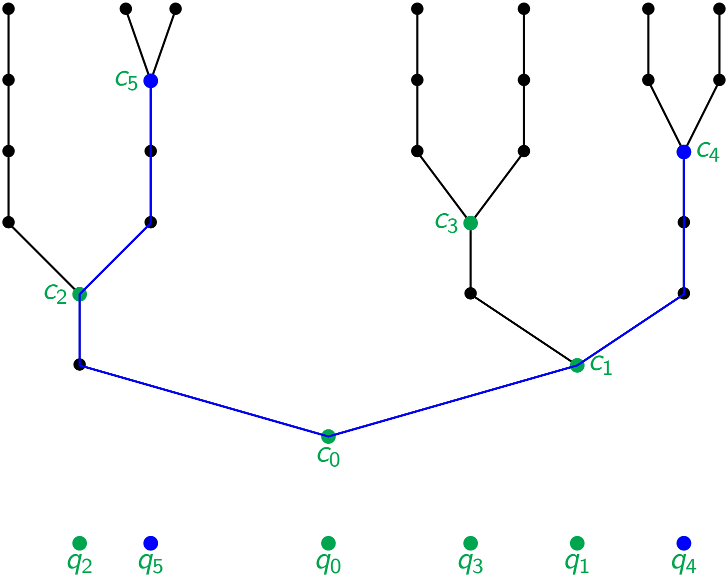
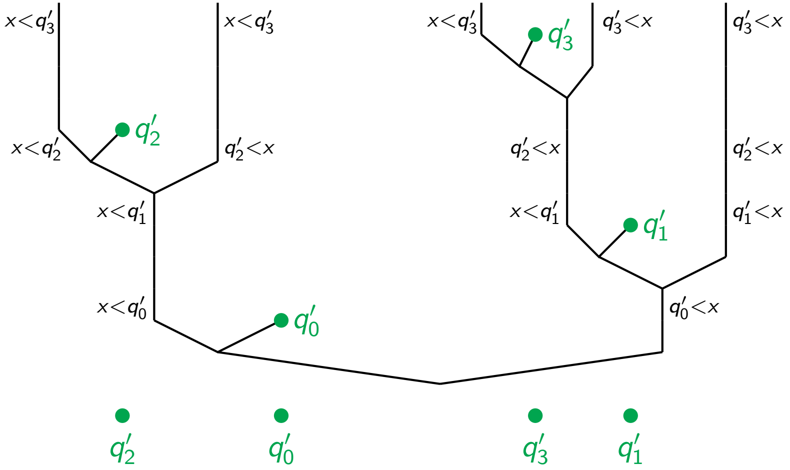
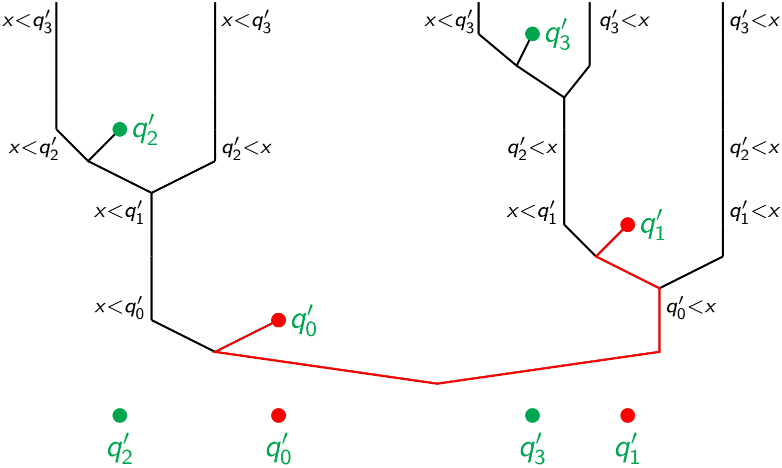


An antichain is diagonal if any two nodes in its meet closure have different lengths.



Devlin's Diagonal Antichains and Exact Degrees



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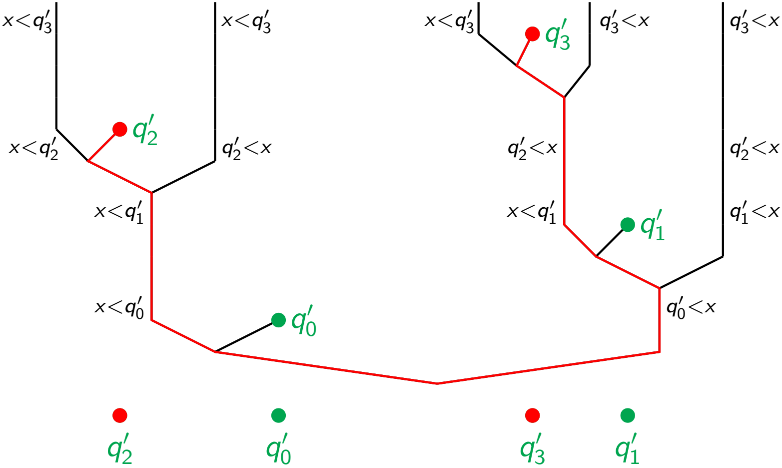

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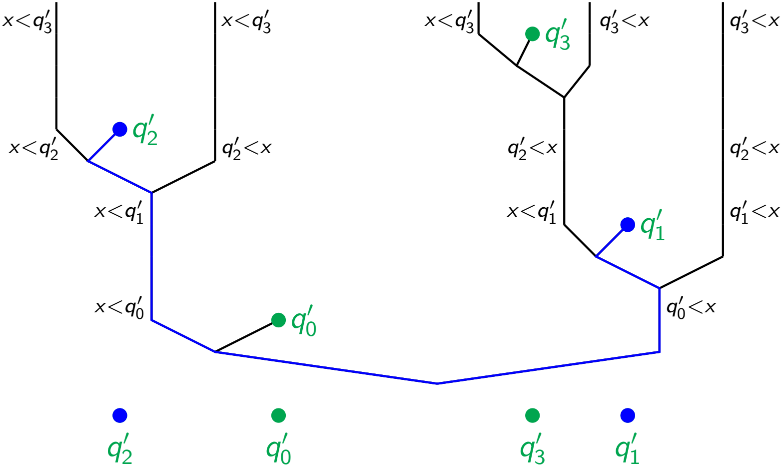

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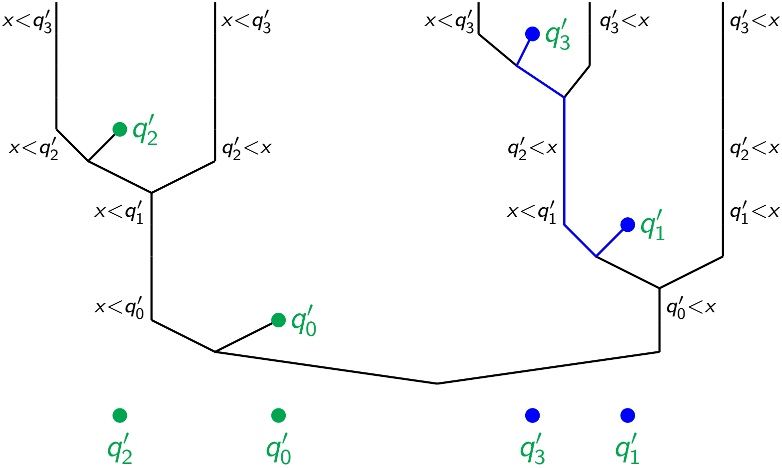


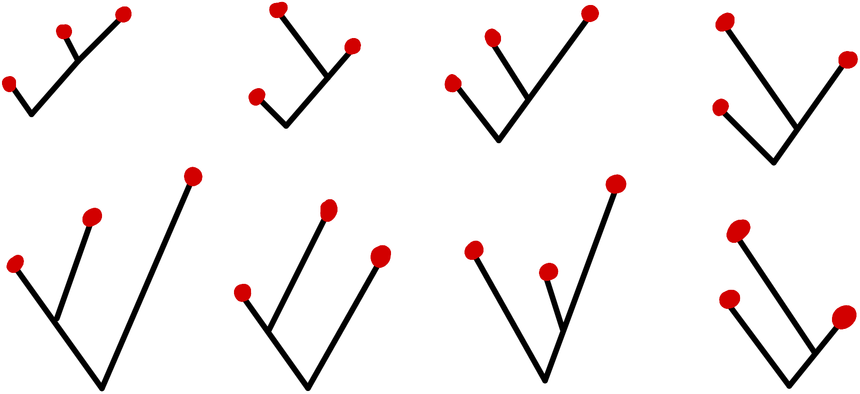
Devlin's Diagonal Antichains and Exact Degrees

Theorem (Devlin, 1979)

For each $k \geq 1$, given any coloring of $[\mathbb{Q}]^k$ into finitely many colors, there is a subset $\mathbb{Q}' \subseteq \mathbb{Q}$ forming a dense linear order such that the k-element subsets of $\mathbb{Q}'$ take at most $T(k) = (2k-1)!c_{2k-1}$ colors, where c_n is from the tangent function $\tan(x) = \sum_{n=0}^{\infty} c_n x^n$.

Devlin types for triples of Rationals

16 Diaries



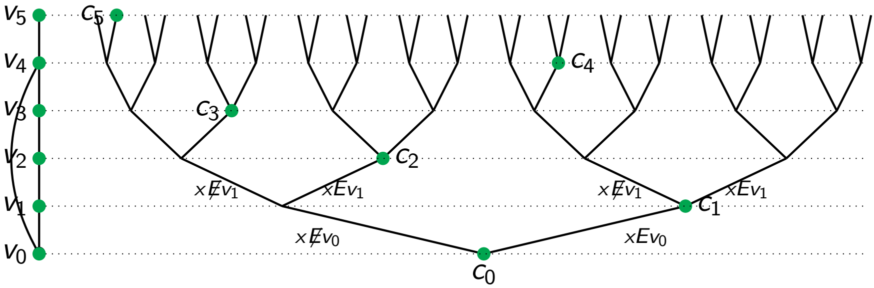
and their mirror images.

Third Ingredient for Rado graph (and other FAP classes)

For the Rado graph, a Third Ingredient is involved in big Ramsey degrees:

passing types encode relations as a longer coding node ‘passes by’ a shorter one.

Coding Tree of 1-types for the Rado Graph, $\mathcal{R}$



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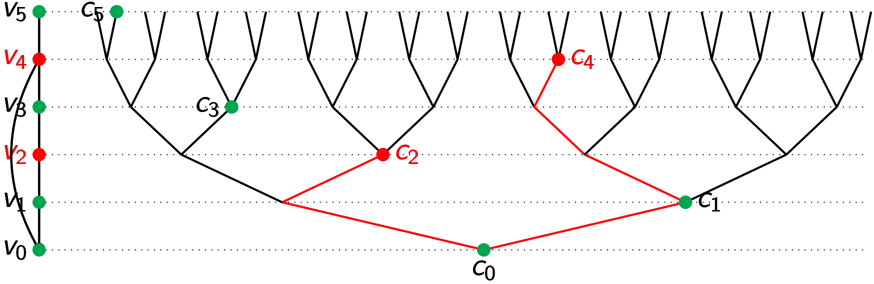
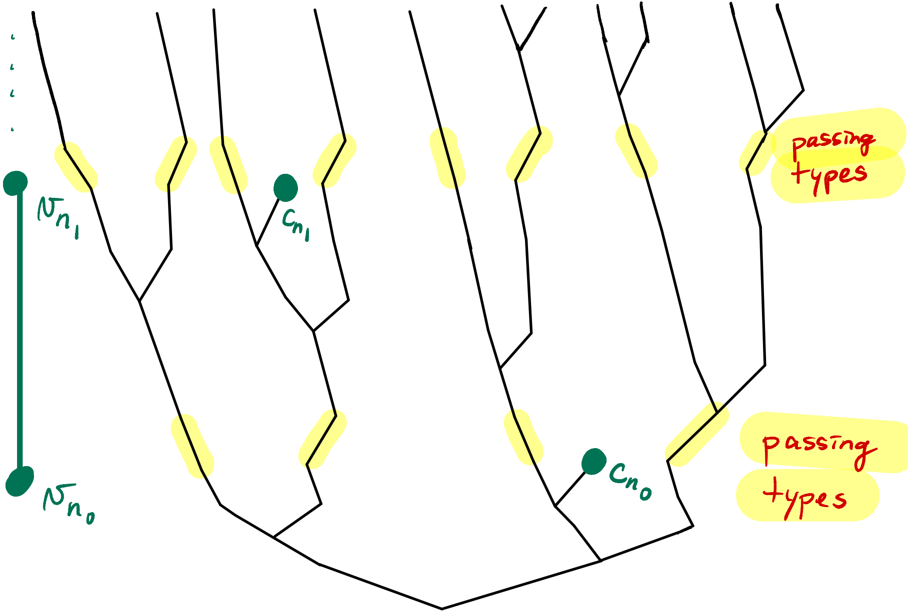


Diagonal Antichain encoding the Rado graph



Big Ramsey degrees of the Rado graph

Theorem (Laflamme–Sauer–Vuksanovic, 2006)

The big Ramsey degree of a finite graph **A** *inside the Rado graph is exactly the number of diagonal antichains encoding a copy of* **A**.

Moreover, the same holds for all unrestricted structures in finitely many binary relations.

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Moreover, the same holds for all unrestricted structures in finitely many binary relations.

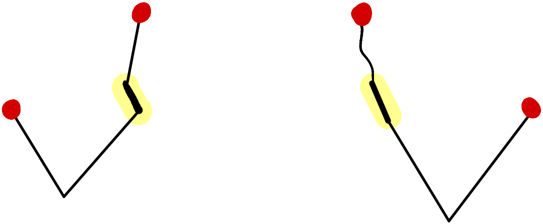
Examples of other unrestricted structures:

- The random directed graph
- Superposition of the Rado graph and a random directed graph

Diaries for BRD of Edges and Non-edges in $\mathcal{R}$

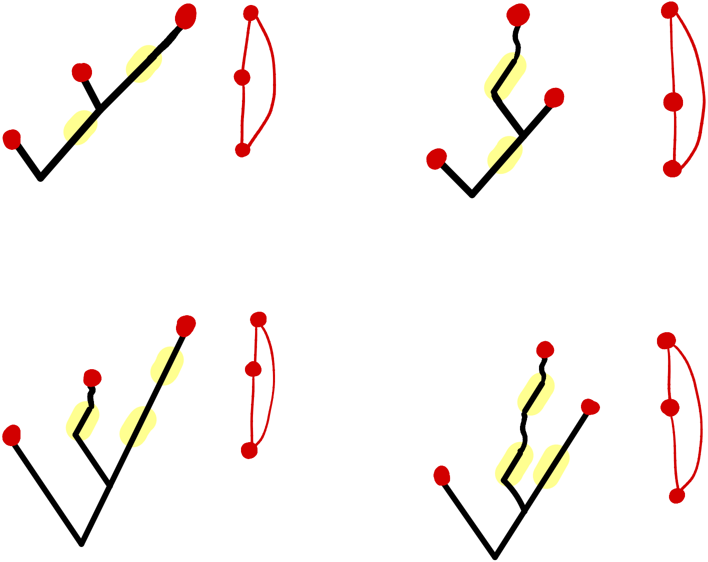
Edges

Non-Edges



Diaries for BRD of Triangles in $\mathcal{R}$

16 Diaries



Remarks

- Classic methods for finding upper bounds for big Ramsey degrees in $\mathbb{Q}$ and Rado graph use Milliken's Ramsey Theorem for Trees. (tomorrow)

- The Halpern-Läuchli Theorem forms the pigeonhole for the proof of Milliken's Theorem.

This aided the development of BRD for free amalgamation classes with forbidden substructures.

V. The Halpern-Läuchli Theorem and Harrington's 'forcing proof'.

Halpern-Läuchli Theorem - strong tree version

Notation:

$$\bigotimes_{i<d} T_i := \bigcup_{n<\omega} \prod_{i<d} T_i(n)$$

Theorem (Halpern-Läuchli, 1966)

Let $T_i \subseteq \omega^{<\omega}$, $i < d$, be finitely branching trees with no terminal nodes. Given a coloring $\chi : \bigotimes_{i<d} T_i \to 2$, there are strong subtrees $S_i \leq T_i$ with nodes of the same lengths such that χ is constant on $\bigotimes_{i<d} S_i$.

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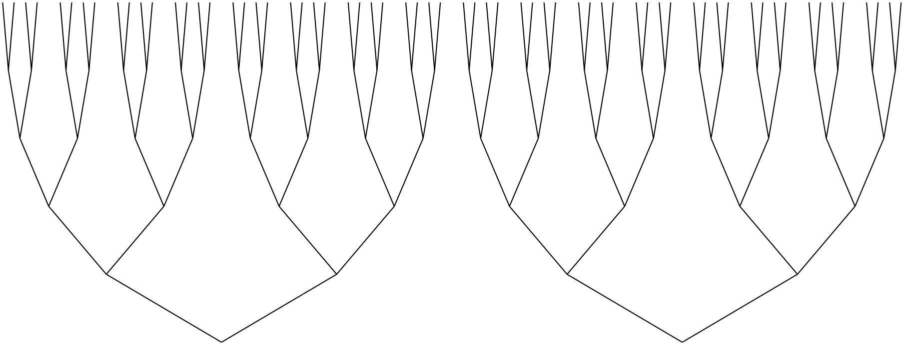
HL was distilled as a key lemma in the proof that the Boolean Prime Ideal Theorem is strictly weaker than the Axiom of Choice over ZF. (Halpern-Lévy, 1971)

Example: Coloring $T_0 \otimes T_1$



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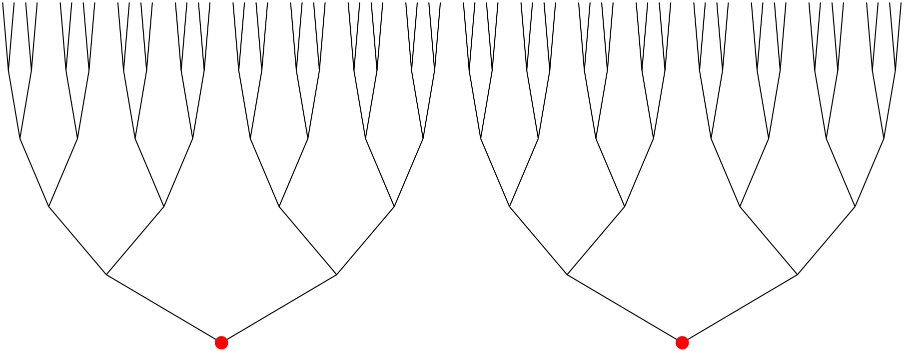

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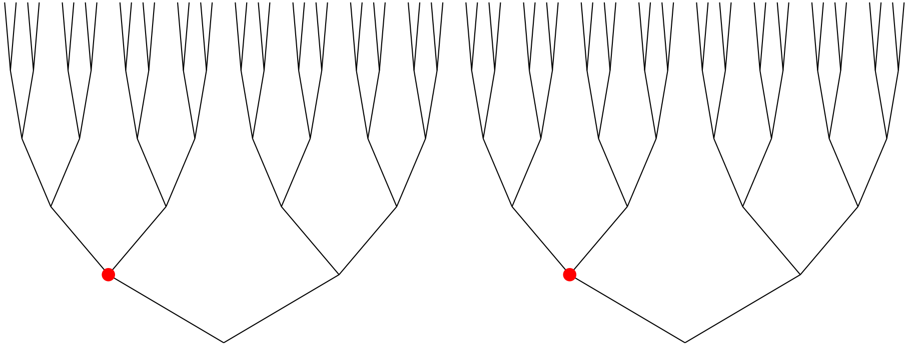

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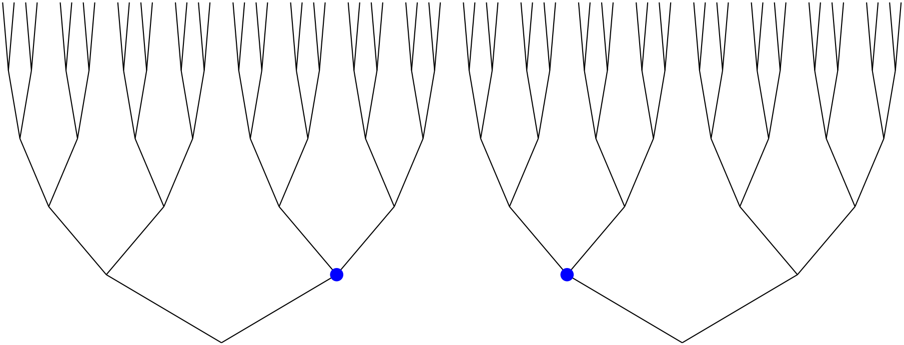

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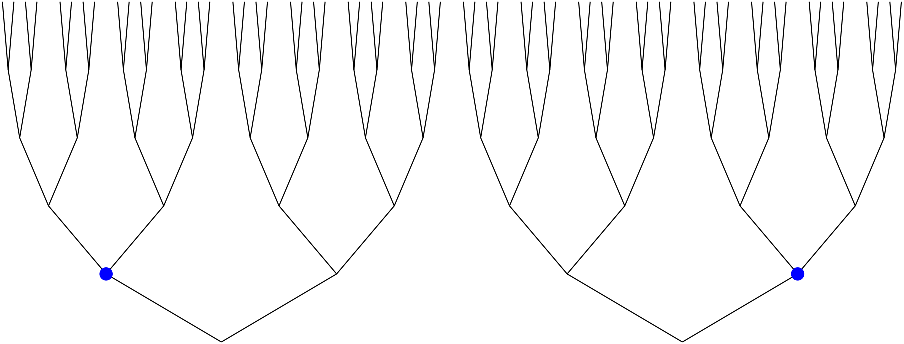

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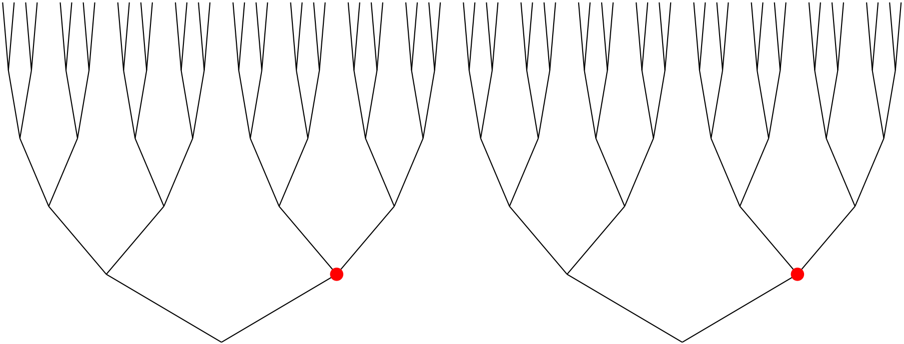

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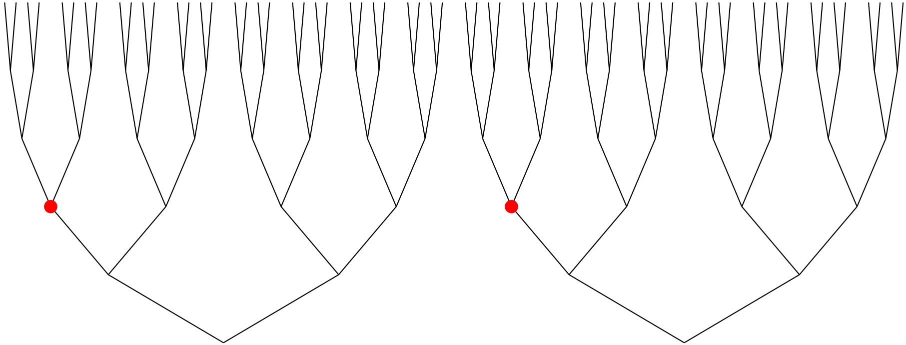

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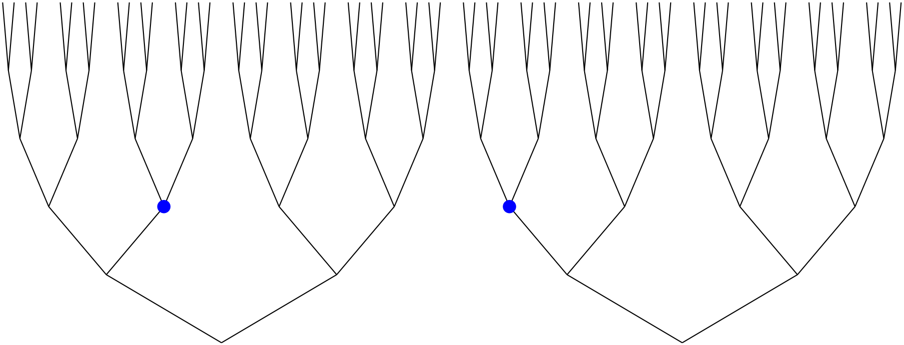

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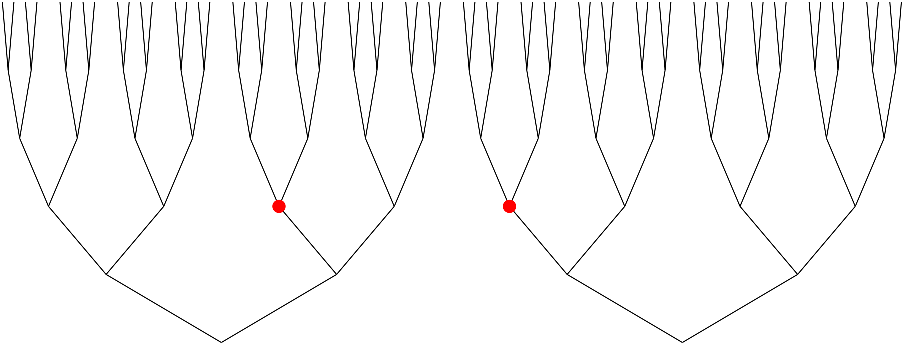

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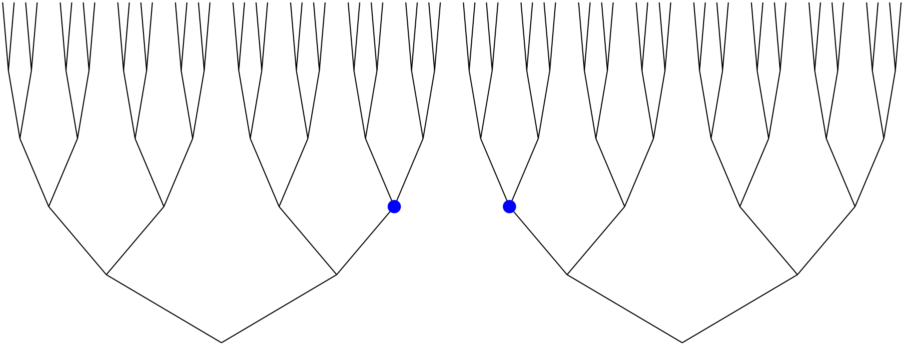

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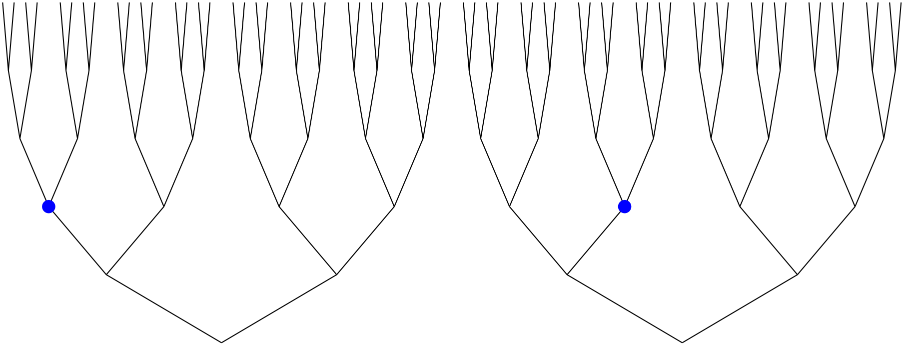

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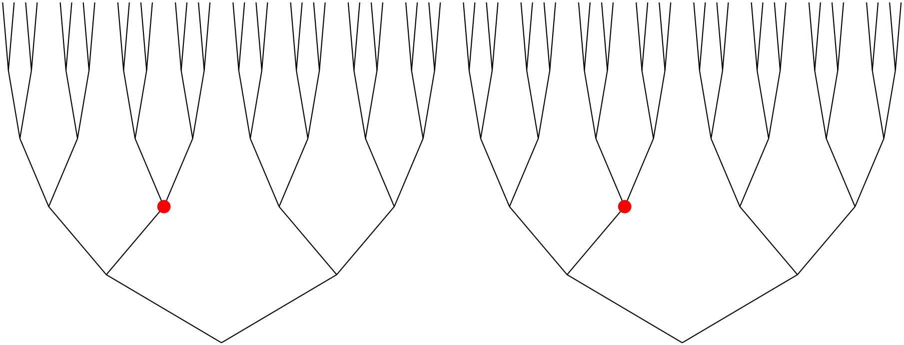
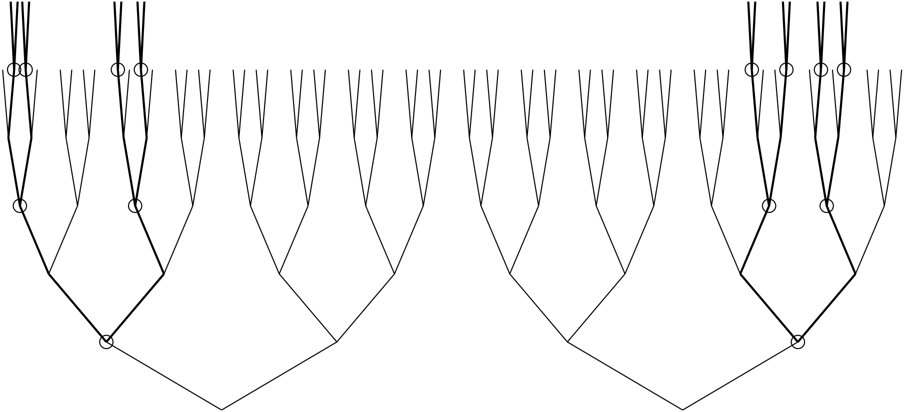


HL gives $S_i \subseteq T_i$ with one color on $S_0 \otimes S_1$



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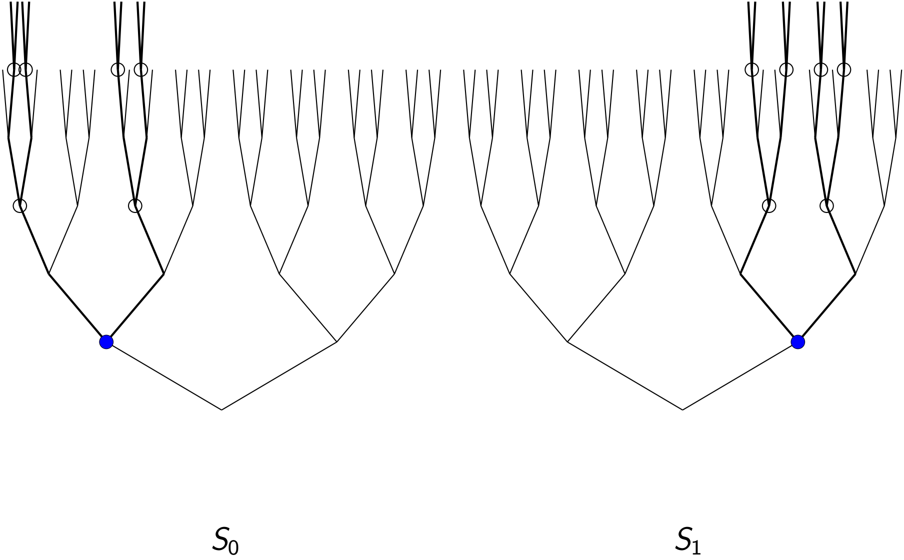

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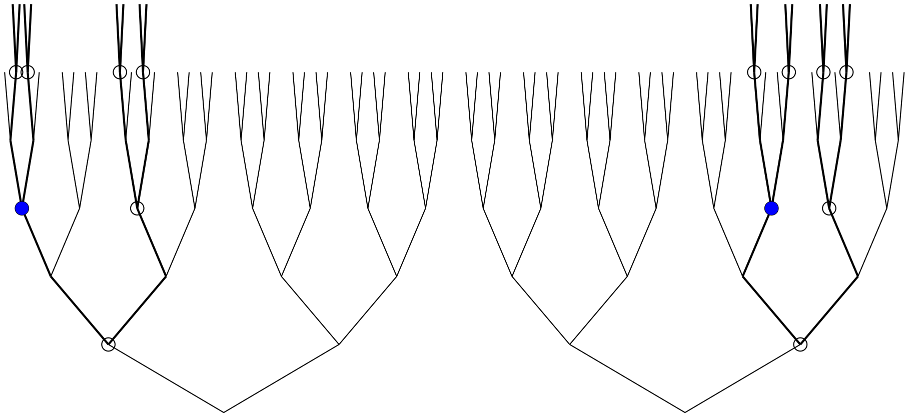

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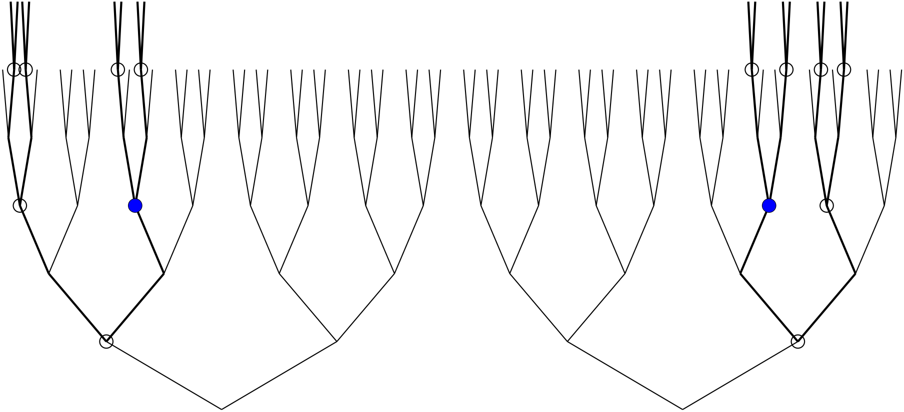

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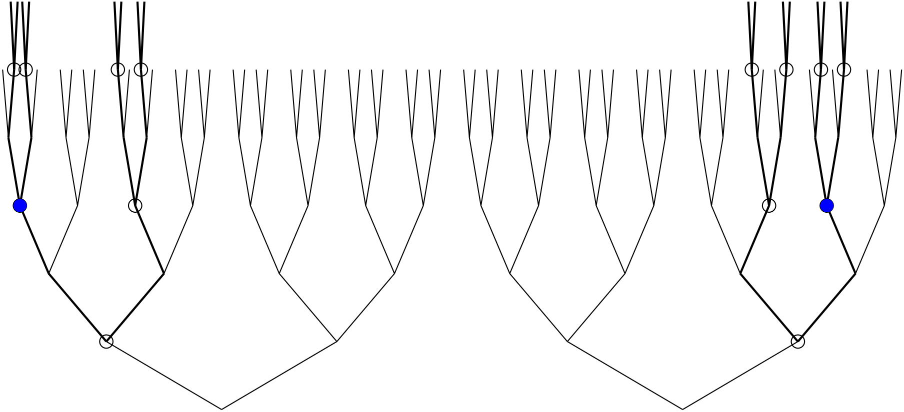

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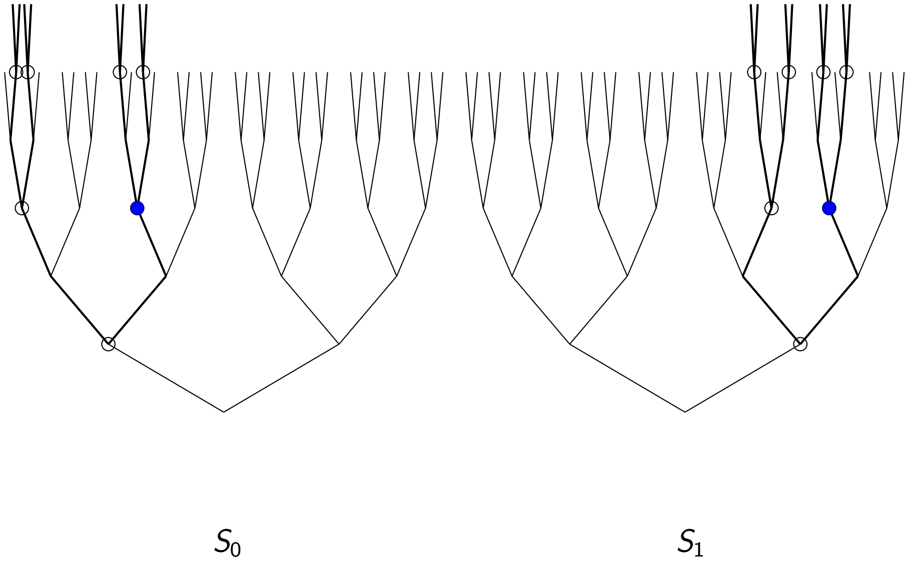

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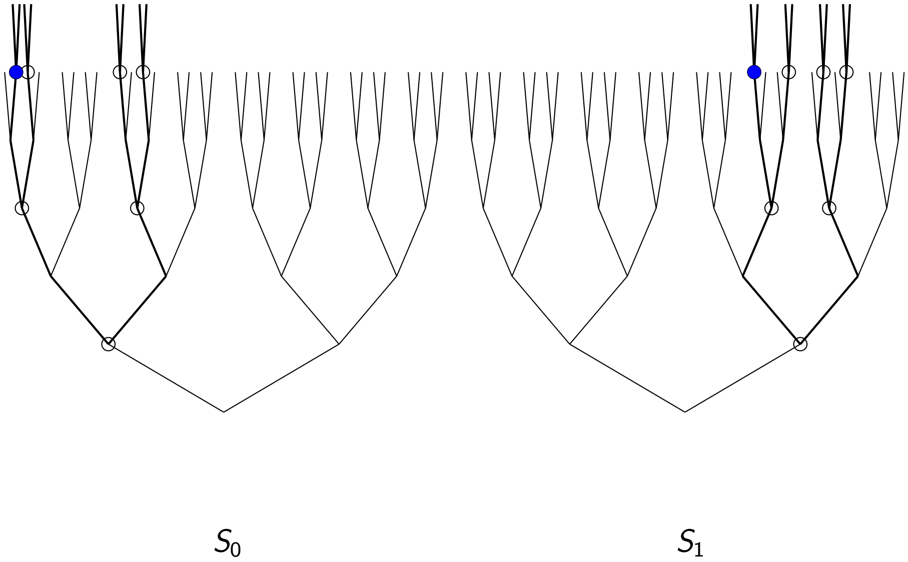

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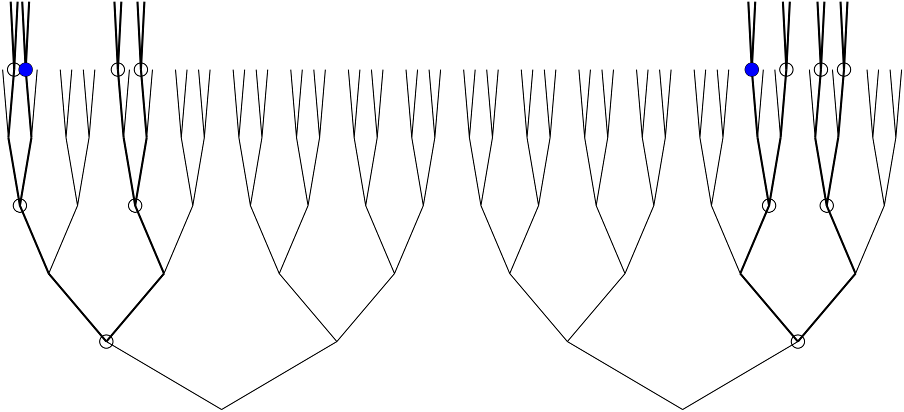

S_0 S_1

An Application of HL to Products of Rationals

Theorem (Laver, 1984)

Given $d < \omega$ and a coloring of $\mathbb{Q}^d$ into finitely many colors, there are $X_i \subseteq \mathbb{Q}$, $i < d$, isomorphic to $\mathbb{Q}$ such that $X_0 \times \cdots \times X_{d-1}$ takes at most $d!$ many colors.

Harrington's 'Forcing' Proof of Halpern-Läuchli Theorem

Harrington devised a proof of the Halpern–Läuchli Theorem that uses forcing methods to do countably many searches for finite objects.

This is NOT an absoluteness proof; no generic extensions involved.

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Thanks to Laver for an outline of this proof in 2011!

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Fix $d \geq 2$ and let $T_i = 2^{<\omega}$ $(i < d)$ be finitely branching trees with no terminal nodes. Fix a coloring $c : \bigotimes_{i<d} T_i \to 2$.

Let $\kappa = \beth_{2d}$. Then $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$.

$\mathbb{P} =$ Cohen forcing adding κ new branches to each tree T_i, $i < d$.

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$\mathbb{P} =$ Cohen forcing adding κ new branches to each tree T_i, $i < d$.

$\mathbb{P}$ is the set of functions p of the form

$$p : d \times \vec{\delta}_p \to \bigcup_{i<d} T_i \restriction \ell_p$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$, $\ell_p < \omega$, and $\forall i < d$, $\{p(i,\delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \restriction \ell_p$.

$q \leq p$ iff $\ell_q \geq \ell_p$, $\vec{\delta}_q \supseteq \vec{\delta}_p$, and $\forall (i,\delta) \in d \times \vec{\delta}_p$, $q(i,\delta) \supseteq p(i,\delta)$.

Harrington's 'Forcing' Proof: Set-up for the Ctbl Coloring

For $i < d$, $\alpha < \kappa$, $\dot{b}_{i,\alpha}$ denotes the α-th generic branch in T_i.

Let $\dot{\mathcal{U}}$ be a $\mathbb{P}$-name for a non-principal ultrafilter on ω.

For $\vec{\alpha} = \langle \alpha_0, \dots, \alpha_{d-1} \rangle \in [\kappa]^d$, let $\dot{b}_{\vec{\alpha}} := \langle \dot{b}_{0,\alpha_0}, \dots, \dot{b}_{d-1,\alpha_{d-1}} \rangle$.

For $\vec{\alpha} \in [\kappa]^d$, take some $p_{\vec{\alpha}} \in \mathbb{P}$ with $\vec{\alpha} \subseteq \vec{\delta}_{p_{\vec{\alpha}}}$ such that

1. $p_{\vec{\alpha}}$ decides an $\varepsilon_{\vec{\alpha}} \in 2$ s.t. $p_{\vec{\alpha}} \Vdash c(\dot{b}_{\vec{\alpha}} \restriction \ell) = \varepsilon_{\vec{\alpha}}$ for $\dot{\mathcal{U}}$ many ℓ,
2. $c(\{p_{\vec{\alpha}}(i, \alpha_i) : i < d\}) = \varepsilon_{\vec{\alpha}}$.

For $\vec{\theta} \in [\kappa]^{2d}$ and $\iota : 2d \to 2d$, let

$$\vec{\alpha} = (\theta_{\iota(0)}, \theta_{\iota(2)}, \ldots, \theta_{\iota(2d-2))}) \text{ and } \vec{\beta} = (\theta_{\iota(1)}, \theta_{\iota(3)}, \ldots, \theta_{\iota(2d-1)}).$$

Define
$$\begin{aligned} f(\iota, \vec{\theta}) = \langle \iota, \varepsilon_{\vec{\alpha}}, k_{\vec{\alpha}}, &\langle\langle p_{\vec{\alpha}}(i, \delta_{\vec{\alpha}}(j)) : j < k_{\vec{\alpha}}\rangle : i < d\rangle, \\ &\langle\langle i, j\rangle : i < d,\ j < k_{\vec{\alpha}}, \delta_{\vec{\alpha}}(j) = \alpha_i\rangle, \\ &\langle\langle j, k\rangle : j < k_{\vec{\alpha}},\ k < k_{\vec{\beta}},\ \delta_{\vec{\alpha}}(j) = \delta_{\vec{\beta}}(k)\rangle\rangle, \end{aligned}$$

where $k_{\vec{\alpha}} = |\vec{\delta}_{p_{\vec{\alpha}}}|$, and $\langle \delta_{\vec{\alpha}}(j) : j < k_{\vec{\alpha}}\rangle$ enumerates $\vec{\delta}_{p_{\vec{\alpha}}}$.

Harrington's 'Forcing' Proof: The Countable Coloring

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Define $f(\vec{\theta}) = \langle f(\iota, \vec{\theta}) : \iota \in \mathcal{I}\rangle$.

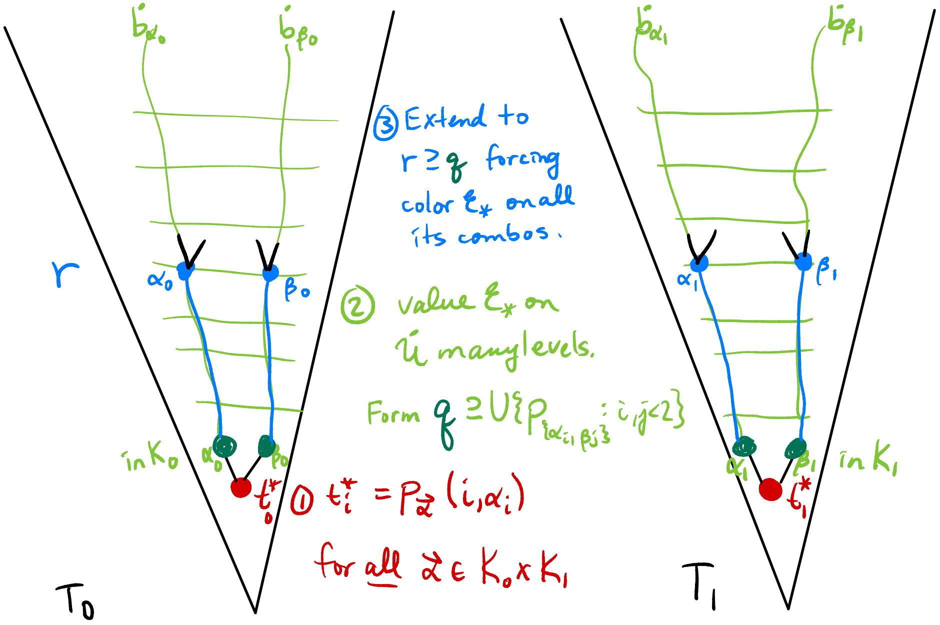
Harrington's 'Forcing' Proof: Set of compatible conditions

$\kappa \to (\aleph_1)^{2d}_{\aleph_0}$ implies $\exists H \in [\kappa]^{\aleph_1}$ homogeneous for f.

Take $K_i \in [H]^{\aleph_0}$ where $K_0 < \cdots < K_{d-1}$ and let $K := \bigcup_{i<d} K_i$.

Main Lemma. $\{p_{\vec{\alpha}} : \vec{\alpha} \in \prod_{i<d} K_i\}$ is compatible.

Harrington's 'Forcing' Proof: The Construction (d=2)



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- Ideas in this proof plus the idea of starting with coding trees of 1-types (rather than regularly branching trees) opened the door to proving the Henson graphs have finite big Ramsey degrees, which inspired an expansion of results and methods.

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- In their AMS Memoirs book (2023), Anglès d'Auriac, Cholak, Dzhafarov, Monin, and Patey, the Halpern-Läuchli Theorem is computably true and admits strong cone avoidance.