Ramsey Theory on Infinite Structures, Part II

Natasha Dobrinen

University of Notre Dame

ASL North American Annual Meeting
UC Irvine, March 25–29, 2023

With grateful acknowledgment of research support from
NSF grants DMS-1301665, 1600781, 1901753

Outline: Day 2

I. Big Ramsey Degree Characterizations and Methods

   (a) $\mathbb{Q}$, Rado, unrestricted: Milliken's Strong Tree Theorem
   (b) SDAP^+ and FAP Structures: Forcing on Coding Trees
   (c) Generic Poset: Parameter Words

II. Infinite-dimensional Ramsey Theory on ω

III. Infinite-dimensional Structural Ramsey Theory

IV. More Directions

Recall from yesterday:

Big Ramsey Degrees

Let $\mathbf{K}$ be an infinite structure.

$\mathbf{K}$ has **finite big Ramsey degrees** if for each finite $\mathbf{A} \le \mathbf{K}$, $\exists T$ such that $\forall r$, $\forall\ \chi : \binom{\mathbf{K}}{\mathbf{A}} \to r$, $\exists \mathbf{K}' \in \binom{\mathbf{K}}{\mathbf{K}}$ such that $|\chi \restriction \binom{\mathbf{K}'}{\mathbf{A}}| \le T$.

The **big Ramsey degree** of $\mathbf{A}$ in $\mathbf{K}$, $T(\mathbf{A})$, is the least such T.

I. Big Ramsey Degree Characterizations and Methods

BRD Characterizations for $\mathbb{Q}$ and Rado graph

(1) Enumerate the universe; induce a coding tree of 1-types.

(2) Diagonal antichain representing $\mathbf{K}$.

(3) Passing types encode binary relations in $\mathbf{K}$.

BRD Characterizations for $\mathbb{Q}$ and Rado graph

(1) Enumerate the universe; induce a coding tree of 1-types.

(2) Diagonal antichain representing $\mathbf{K}$.

(3) Passing types encode binary relations in $\mathbf{K}$.

- $\mathbb{Q}$ uses (1) and (2). (3) is redundant as the order relation is always determined at lower levels by transitivity.
- Rado graph uses (1)–(3).

BRD Characterizations for $\mathbb{Q}$ and Rado graph

(1) Enumerate the universe; induce a coding tree of 1-types.

(2) Diagonal antichain representing $\mathbf{K}$.

(3) Passing types encode binary relations in $\mathbf{K}$.

- $\mathbb{Q}$ uses (1) and (2). (3) is redundant as the order relation is always determined at lower levels by transitivity.
- Rado graph uses (1)–(3).

Terminology: Devlin types ($\mathbb{Q}$), similarity types [LSV], (diagonal) diaries [BCDHKVZ] and following

Yesterday I went through coding trees of 1-types.

Yesterday I went through coding trees of 1-types.

Coding trees were developed in 2015 to handle the triangle-free Henson graph. In hindsight they were coding trees of 1-types.

Yesterday I went through coding trees of 1-types.

Coding trees were developed in 2015 to handle the triangle-free Henson graph. In hindsight they were coding trees of 1-types.

Coding trees of 1-types were first made explicit in 2020 in [Coulson–D.–Patel], including the coding tree of 1-types for $\mathbb{Q}$.

Yesterday I went through coding trees of 1-types.

Coding trees were developed in 2015 to handle the triangle-free Henson graph. In hindsight they were coding trees of 1-types.

Coding trees of 1-types were first made explicit in 2020 in [Coulson–D.–Patel], including the coding tree of 1-types for $\mathbb{Q}$.

Now we will go through the classic methodology.

I. Big Ramsey Degree Characterizations and Methods

(a) Results using Milliken's Theorem

Classic methodology: [Laflamme, Sauer, Vuksanovic, 2006]

1. Tree $k^{<\omega}$ represents a universal structure.
   (a) Represent $\mathbb{Q}$ by $2^{<\omega}$ with lexicographic order.
   (b) Represent a universal graph by $2^{<\omega}$.
   (c) Represent a universal structure for Age($\mathbf{K}$) by $k^{<\omega}$ for an unrestricted structure $\mathbf{K}$ with n binary relations, where $k = 2^n$.

Classic methodology: [Laflamme, Sauer, Vuksanovic, 2006]

1. Tree $k^{<\omega}$ represents a universal structure.
   (a) Represent $\mathbb{Q}$ by $2^{<\omega}$ with lexicographic order.
   (b) Represent a universal graph by $2^{<\omega}$.
   (c) Represent a universal structure for Age($\mathbf{K}$) by $k^{<\omega}$ for an unrestricted structure $\mathbf{K}$ with n binary relations, where $k = 2^n$.

2. Use Milliken's Ramsey theorem for strong subtrees and envelopes to obtain upper bounds for BRD.

Classic methodology: [Laflamme, Sauer, Vuksanovic, 2006]

1. Tree $k^{<\omega}$ represents a universal structure.
   (a) Represent $\mathbb{Q}$ by $2^{<\omega}$ with lexicographic order.
   (b) Represent a universal graph by $2^{<\omega}$.
   (c) Represent a universal structure for Age($\mathbf{K}$) by $k^{<\omega}$ for an unrestricted structure $\mathbf{K}$ with n binary relations, where $k = 2^n$.

2. Use Milliken's Ramsey theorem for strong subtrees and envelopes to obtain upper bounds for BRD.

3. Construct a diagonal antichain inside the infinite binary branching tree which represents a subcopy of $\mathbf{K}$.

Classic methodology: [Laflamme, Sauer, Vuksanovic, 2006]

1. Tree $k^{<\omega}$ represents a universal structure.
   (a) Represent $\mathbb{Q}$ by $2^{<\omega}$ with lexicographic order.
   (b) Represent a universal graph by $2^{<\omega}$.
   (c) Represent a universal structure for Age($\mathbf{K}$) by $k^{<\omega}$ for an unrestricted structure $\mathbf{K}$ with n binary relations, where $k = 2^n$.

2. Use Milliken's Ramsey theorem for strong subtrees and envelopes to obtain upper bounds for BRD.

3. Construct a diagonal antichain inside the infinite binary branching tree which represents a subcopy of $\mathbf{K}$.

4. Show that the BRD are exactly characterized via diagonal antichains encoding the structures. (lower bounds)

Milliken's Theorem (special case)

Theorem (Milliken, 1979)

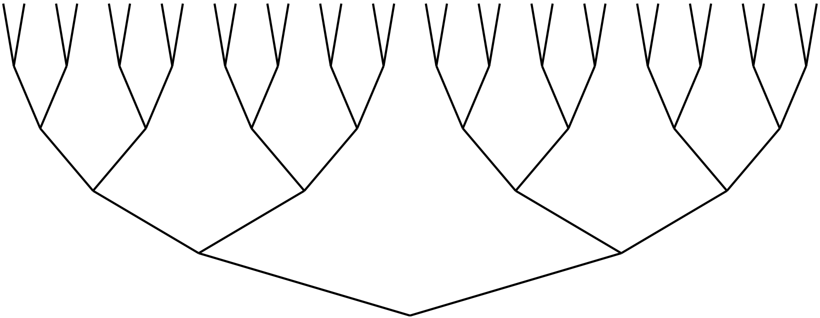
Given $n \geq 1$ and a coloring of all n-strong subtrees of $2^{<\omega}$ into finitely many colors, there is an infinite strong subtree of $2^{<\omega}$ in which all n-strong subtrees have the same color.

A subtree T of $2^{<\omega}$ is **n-strong** if it is isomorphic to the tree 2^{n-1}.

Milliken's Theorem (special case)

Theorem (Milliken, 1979)

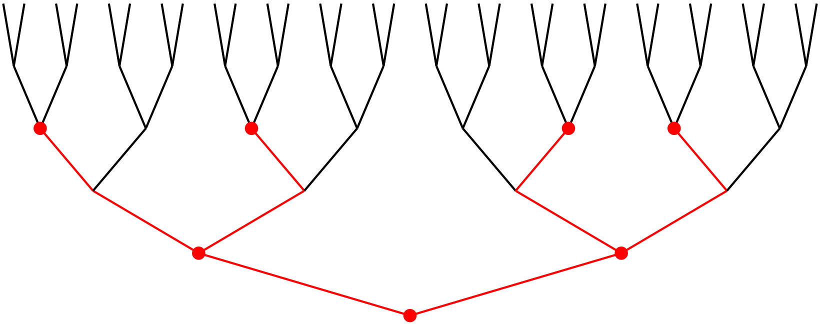
Given $n \geq 1$ and a coloring of all n-strong subtrees of $2^{<\omega}$ into finitely many colors, there is an infinite strong subtree of $2^{<\omega}$ in which all n-strong subtrees have the same color.



Milliken's Theorem (special case)

Theorem (Milliken, 1979)

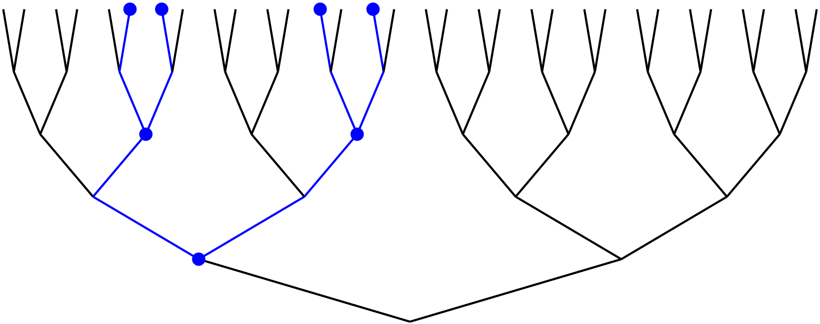
Given $n \geq 1$ and a coloring of all n-strong subtrees of $2^{<\omega}$ into finitely many colors, there is an infinite strong subtree of $2^{<\omega}$ in which all n-strong subtrees have the same color.



Milliken's Theorem (special case)

Theorem (Milliken, 1979)

Given $n \geq 1$ and a coloring of all n-strong subtrees of $2^{<\omega}$ into finitely many colors, there is an infinite strong subtree of $2^{<\omega}$ in which all n-strong subtrees have the same color.



Milliken's Theorem (special case)

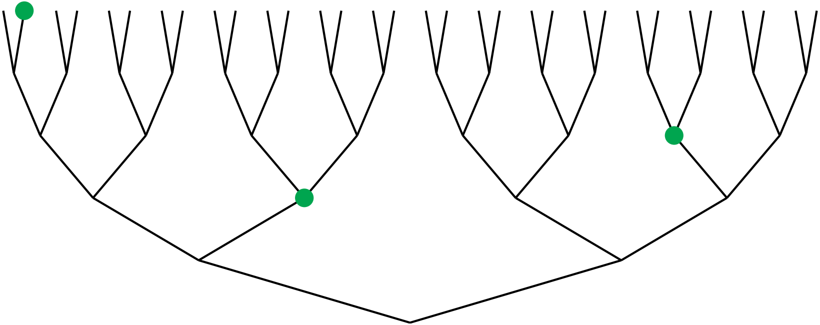
Theorem (Milliken, 1979)

Given $n \geq 1$ and a coloring of all n-strong subtrees of $2^{<\omega}$ into finitely many colors, there is an infinite strong subtree of $2^{<\omega}$ in which all n-strong subtrees have the same color.

Remark.

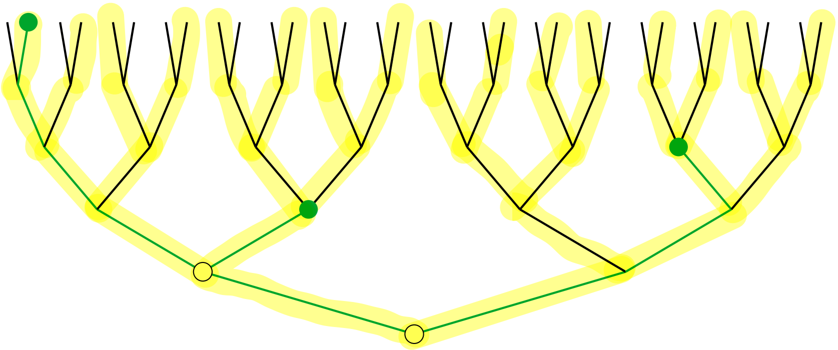
- Halpern–Läuchli Theorem forms the pigeonhole principle in the proof of Milliken's Theorem.

Envelopes of Diagonal Antichains



Envelopes of Diagonal Antichains

Apply Milliken to get one color per diary.



Then pull out diagonal antichain encoding $\mathbb{K}$.

Finite BRD for unrestricted higher arity relations

Theorem (Balko, Chodounský, Hubička, Konečný, Vena, 2022)

The 3-uniform generic hypergraph has finite big Ramsey degrees.

Proof uses product tree Milliken Theorem and a clever encoding of the ternary edge relation via two trees.

Finite BRD for unrestricted higher arity relations

Theorem (Balko, Chodounský, Hubička, Konečný, Vena, 2022)

The 3-uniform generic hypergraph has finite big Ramsey degrees.

Proof uses product tree Milliken Theorem and a clever encoding of the ternary edge relation via two trees.

Theorem (Braunfeld, Chodounský, de Rancourt, Hubička, Kawach, Konečný, 2023+)

Given a countable relational language $\mathcal{L}$ with finitely many relations of every arity > 1, let $\mathcal{K}$ be the Fraïssé class of finite unrestricted $\mathcal{L}$-structures. The Fraïssé limit has finite big Ramsey dgrees.

Proof uses [Laver 1984] Ramsey Theorem for product of infinitely many trees.

I. Big Ramsey Degree Characterizations and Methods

(b) Results using Coding Trees and Forcing Methods

Free Amalgamation Classes with Forbidden Substructures

Problem 11.2 in [KPT 05] asked (among many other things) whether the k-clique-free Henson graphs have finite big Ramsey degrees.

Prior Methods Insufficient for Triangle-Free Graphs

The **Henson graph**, $\mathcal{H}_3$, is the infinite homogeneous triangle-free graph into which every finite triangle-free graph embeds.

Prior Methods Insufficient for Triangle-Free Graphs

The **Henson graph**, $\mathcal{H}_3$, is the infinite homogeneous triangle-free graph into which every finite triangle-free graph embeds.

Previous Results:

- $T(\text{vertex},\mathcal{H}_3) = 1$, Pigeonhole Principle (Komjáth–Rödl, 1986)
- $T(\text{Edge},\mathcal{H}_3) = 2$ (Sauer, 1998)

Milliken's Theorem unable to handle triangle-free graphs.

New developments: how forcing opened new paths

In December 2015 in Cambridge, new ideas came to me:

- Start with the end in sight, and
- Try big machinery first: forcing.

In December 2015 in Cambridge, new ideas came to me:

- Start with the end in sight, and
- Try big machinery first: forcing.

  precursor: Harrington's forcing proof of Halpern-Läuchli.

In December 2015 in Cambridge, new ideas came to me:

- Start with the end in sight, and
- Try big machinery first: forcing.

  precursor: Harrington's forcing proof of Halpern-Läuchli.

1. Coding Trees: Enumerate a Henson graph and use it to determine the correct tree structure.

New developments: how forcing opened new paths

In December 2015 in Cambridge, new ideas came to me:

- Start with the end in sight, and
- Try big machinery first: forcing.

  precursor: Harrington's forcing proof of Halpern-Läuchli.

1. Coding Trees: Enumerate a Henson graph and use it to determine the correct tree structure.
2. A Ramsey Theorem for Coding Trees: Use the set-theoretic method of forcing to do unbounded searches for finite objects.

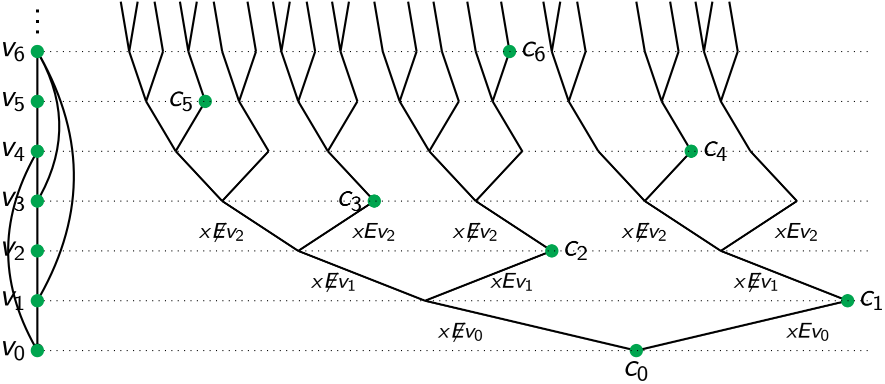
In December 2015 in Cambridge, new ideas came to me:

- Start with the end in sight, and
- Try big machinery first: forcing.

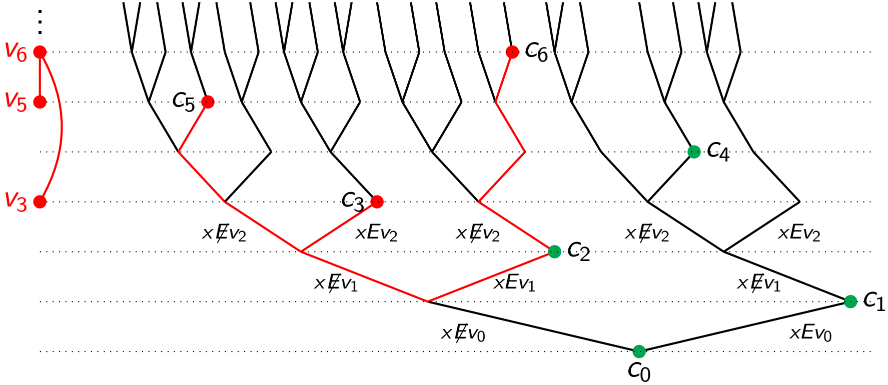
  precursor: Harrington's forcing proof of Halpern-Läuchli.

1. Coding Trees: Enumerate a Henson graph and use it to determine the correct tree structure.
2. A Ramsey Theorem for Coding Trees: Use the set-theoretic method of forcing to do unbounded searches for finite objects.
3. A new notion of envelope.

Coding Tree of 1-types for $\mathcal{H}_3$



Coding Tree of 1-types for $\mathcal{H}_3$



Forcing a Level Set Pigeonhole (HL analogue)

A **level set** is a set of nodes with the same length.

- Fix $\mathbf{K}$. Let $\mathbb{S}$ be its coding tree of 1-types.
- Fix a finite subtree $A \subseteq \mathbb{S}$ and fix a level set X in $\mathbb{T}$ end-extending A.
- Color all copies of X extending A into two colors.
- Build a subtree T representing a subcopy of $\mathbf{K}$ in which all copies of X extending A have the same color.

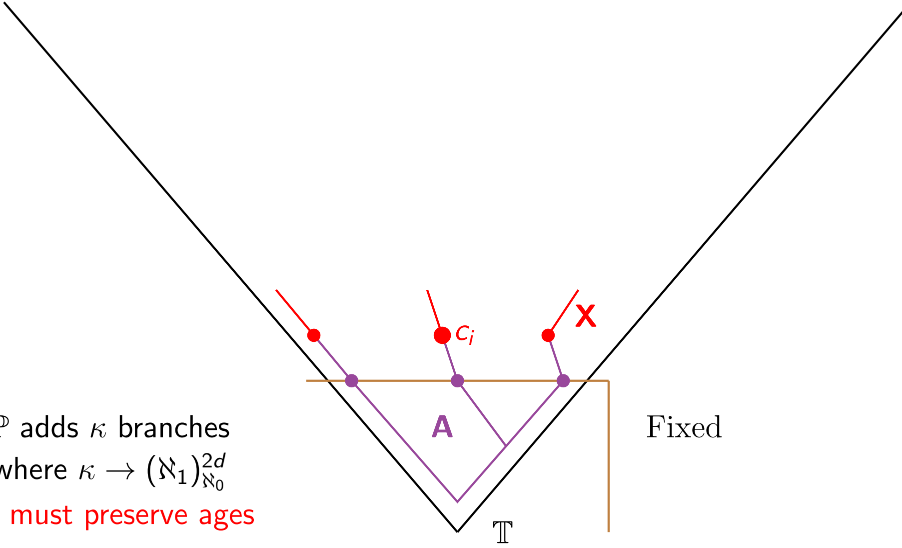
Forcing a Level Set Pigeonhole (HL analogue)

A **level set** is a set of nodes with the same length.

- Fix $\mathbf{K}$. Let $\mathbb{S}$ be its coding tree of 1-types.
- Fix a finite subtree $A \subseteq \mathbb{S}$ and fix a level set X in $\mathbb{T}$ end-extending A.
- Color all copies of X extending A into two colors.
- Build a subtree T representing a subcopy of $\mathbf{K}$ in which all copies of X extending A have the same color.

Three challenges: 1) Figure out the right partial order to force with.
2) Find good starting nodes for building a subtree.
3) Build a subtree encoding $\mathbf{K}$ in which all copies of X have the same color.

Forcing a Level Set Pigeonhole

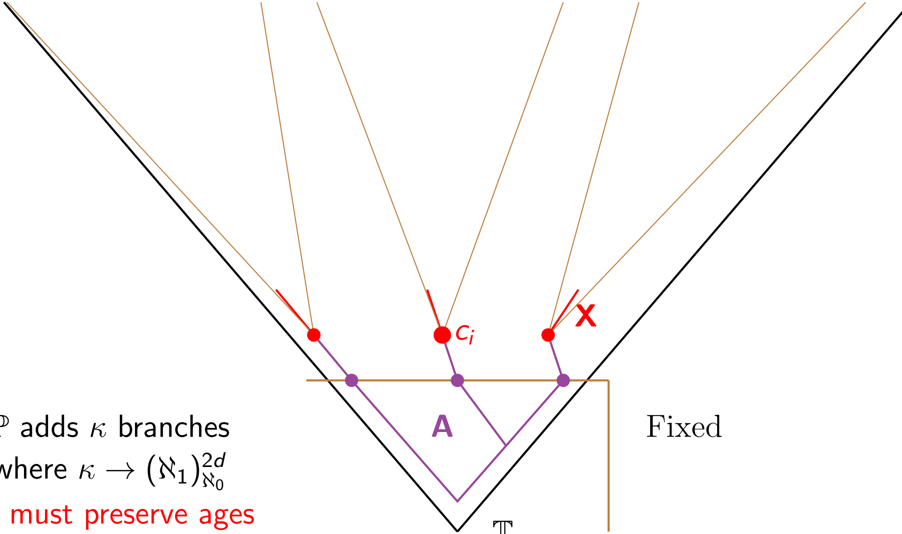


$\mathbb{P}$ adds κ branches

where $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$

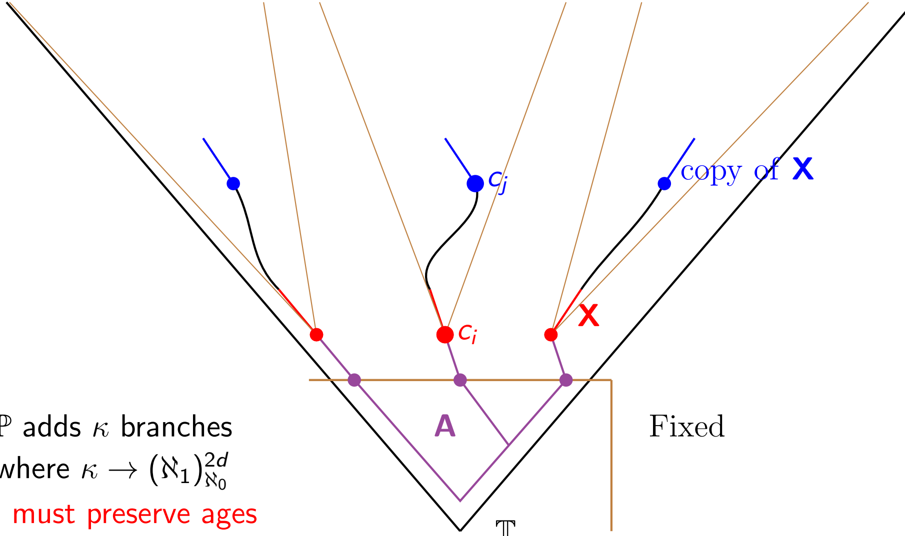
$\leq$ must preserve ages

Forcing a Level Set Pigeonhole



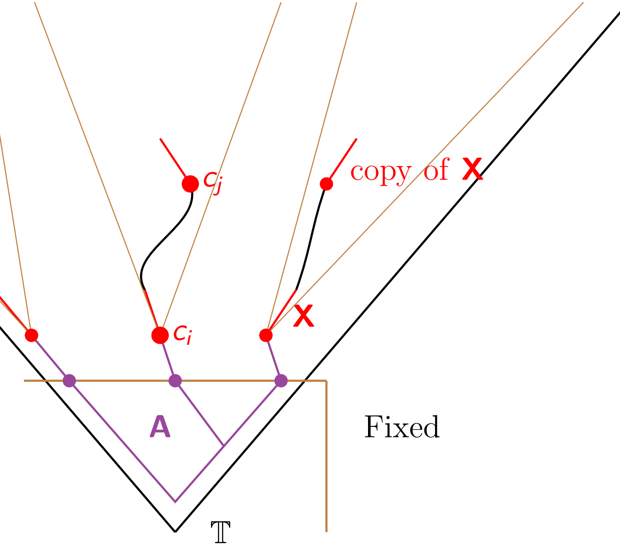
$\mathbb{P}$ adds κ branches
where $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$
$\leq$ must preserve ages

Forcing a Level Set Pigeonhole



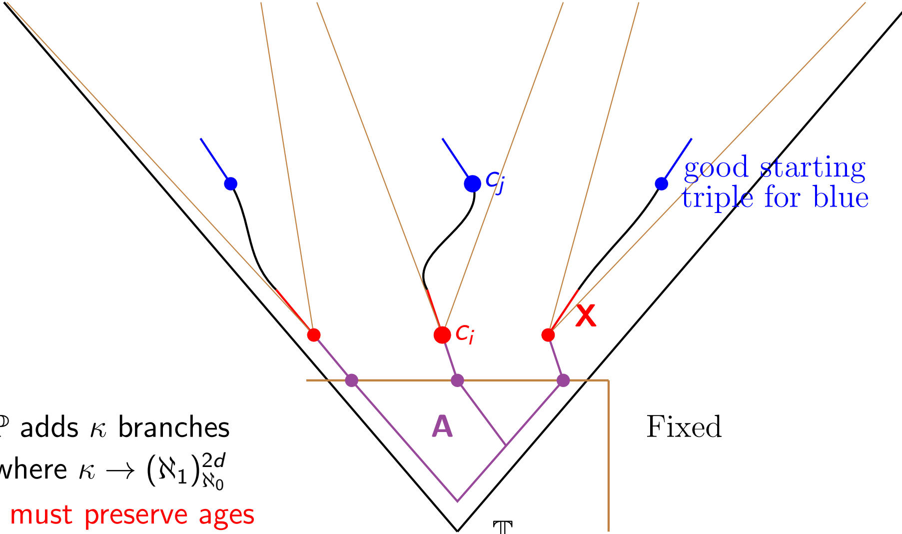
$\mathbb{P}$ adds κ branches
where $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$
$\leq$ must preserve ages

Forcing a Level Set Pigeonhole



$\mathbb{P}$ adds κ branches
where $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$
$\leq$ must preserve ages

Forcing a Level Set Pigeonhole

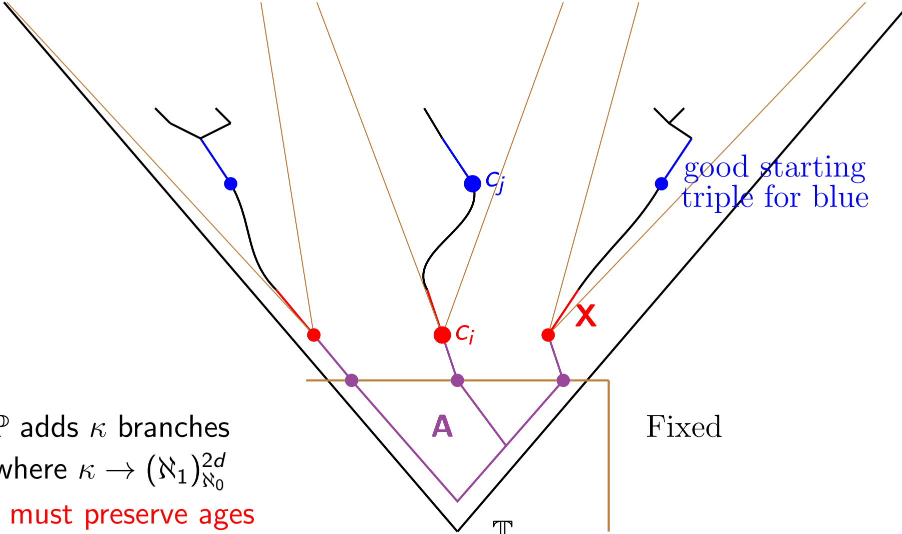


$\mathbb{P}$ adds κ branches

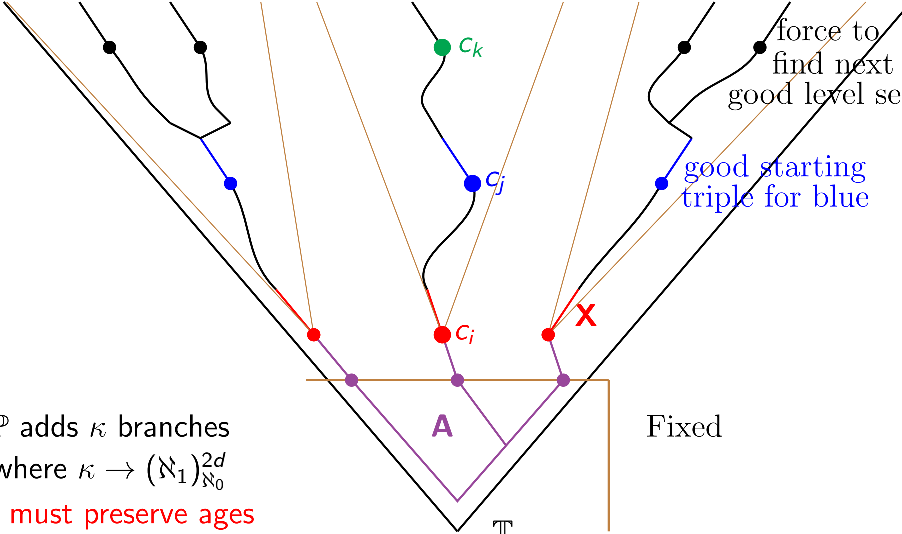
where $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$

$\leq$ must preserve ages

Forcing a Level Set Pigeonhole

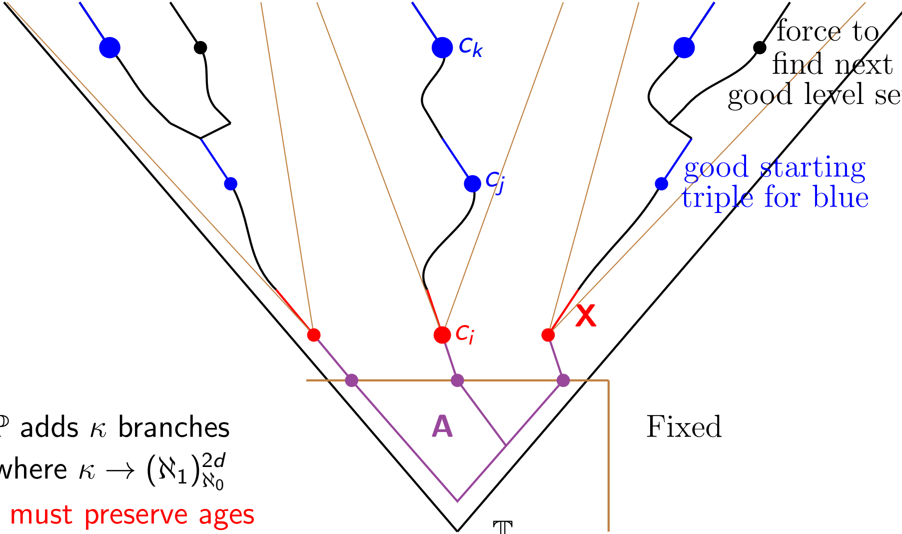


$\mathbb{P}$ adds κ branches
where $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$
$\leq$ must preserve ages



$\mathbb{P}$ adds κ branches
where $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$
$\leq$ must preserve ages

Forcing a Level Set Pigeonhole

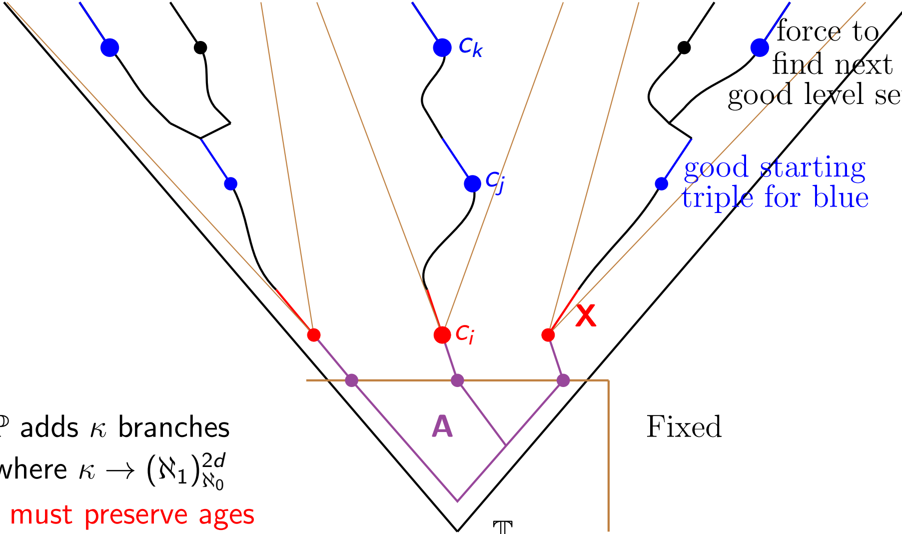


$\mathbb{P}$ adds κ branches

where $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$

$\leq$ must preserve ages

Forcing a Level Set Pigeonhole

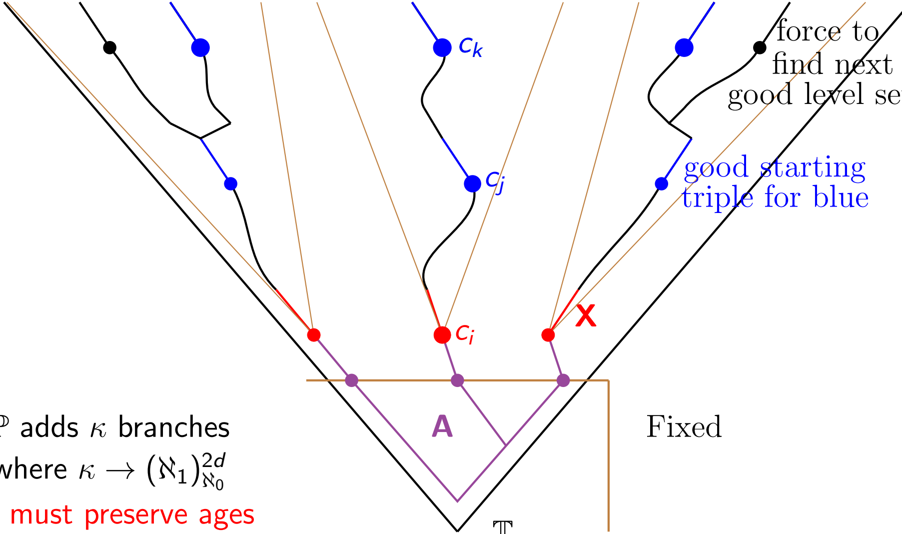


$\mathbb{P}$ adds κ branches

where $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$

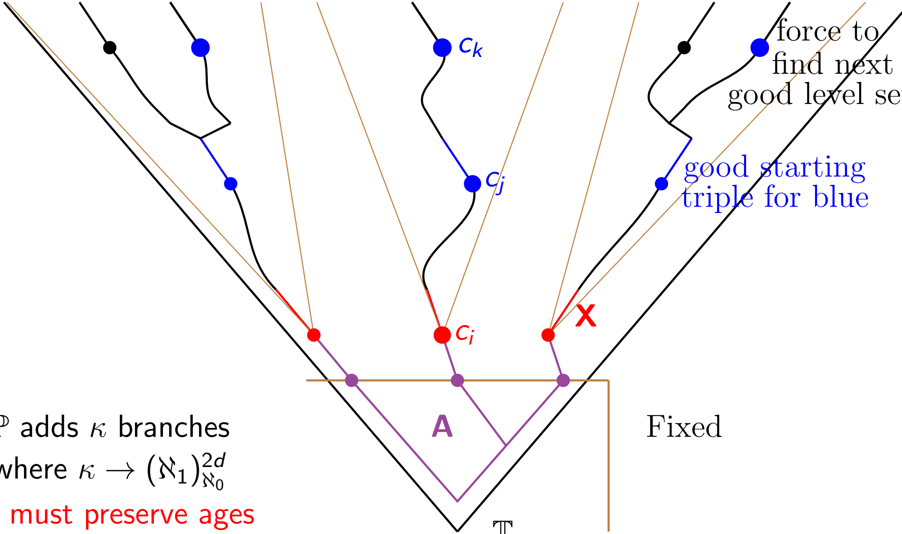
$\leq$ must preserve ages

Forcing a Level Set Pigeonhole



$\mathbb{P}$ adds κ branches
where $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$
$\leq$ must preserve ages

Forcing a Level Set Pigeonhole



$\mathbb{P}$ adds κ branches

where $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$

$\leq$ must preserve ages

Upper bounds for Henson graphs

Theorem (D., JML 2020 and 2022)

The triangle-free and more generally all k-clique-free Henson graphs have finite big Ramsey degrees.

Proofs directly reproduce indivisibility.

Exact BRD for triangle-free Henson graph

A small tweak of the trees in [D.2020] produces exact big Ramsey degrees.

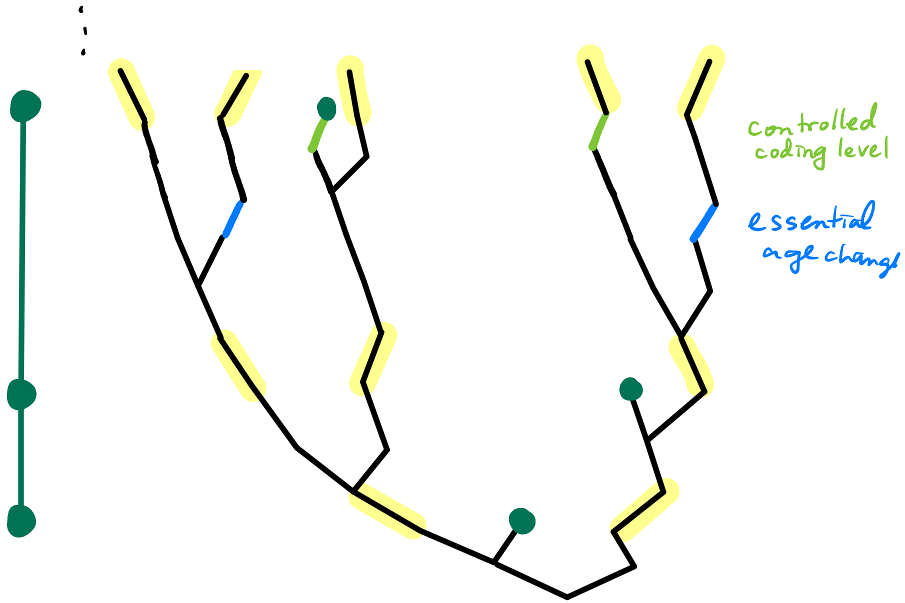
Theorem (D. and independently, Balko, Chodounský, Hubička, Konečný, Vena, Zucker, 2020)

Exact big Ramsey degrees of the triangle-free Henson graph are characterized.

The characterization involves

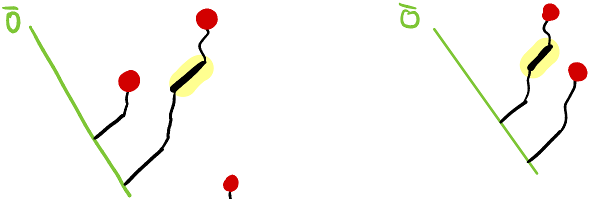
(1) Diagonal antichains;
(2) Controlled age-change levels: first levels of pairs coding of edges with a common vertex in $\mathcal{H}_3$;
(3) Controlled coding levels;
(4) Controlled paths: first level off of leftmost branch.

A Strong (Diagonal) Diary for $\mathcal{H}_3$

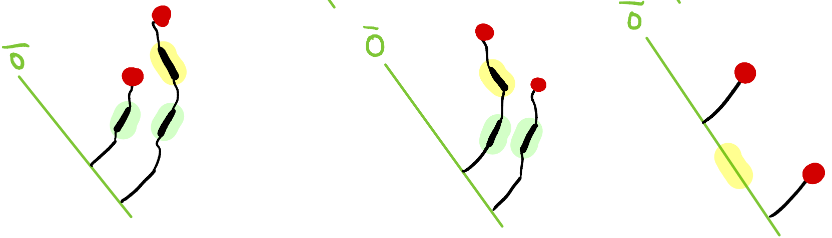


BRD for pairs in Triangle-Free Henson Graph Diaries

Edges



Non-edges



Finitely Constrained Binary FAP Classes

Fix a language $\mathcal{L}$ with finitely many relations of arity at most 2.

An $\mathcal{L}$-structure is **irreducible** if any two vertices are in some relation: e.g., finite clique, finite tournament, triangle with 2 red edges and one blue edge.

Free amalgamation classes are exactly of the form $\mathrm{Forb}(\mathcal{F})$, where $\mathcal{F}$ is a set of finite **irreducible** structures.

Finitely Constrained Binary FAP Classes

Fix a language $\mathcal{L}$ with finitely many relations of arity at most 2.

An $\mathcal{L}$-structure is **irreducible** if any two vertices are in some relation: e.g., finite clique, finite tournament, triangle with 2 red edges and one blue edge.

Free amalgamation classes are exactly of the form $\mathrm{Forb}(\mathcal{F})$, where $\mathcal{F}$ is a set of finite **irreducible** structures.

Theorem (Zucker, 2022)

All finitely constrained binary FAP classes have finite big Ramsey degrees.

BRD of Finitely Constrained Binary FAP Classes

Theorem (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker, 2021+)

The exact big Ramsey degrees of finitely constrained binary FAP classes are characterized by the following:

1. *Diagonal antichains*
2. *Controlled splitting levels*
3. *Controlled age-change levels (essential changes in the class of structures which can be glued above a finite structure to make a member of $\mathcal{K}$)*
4. *Controlled coding levels (reducing the ages of the extending class as much as possible)*
5. *Controlled paths (only matter for non-trivial unary relations)*

An unexpected application of coding trees and forcing to structures which behave like $\mathbb{Q}$ or the Rado graph:

SDAP$^+$ $\Rightarrow$ Simple BRD

Theorem (Coulson–D.–Patel)

Let $\mathcal{L}$ be a finite relational language and let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit satisfying the Substructure Disjoint Amalgamation Property$^+$. Let $\mathbf{K} = Flim(\mathcal{K})$.

I. $\mathbf{K}$ is indivisible.

II. If $\mathcal{L}$ has no relations of arity greater than two, then $\mathbf{K}$ has big Ramsey degrees characterized by diagonal antichains.

This class of structures includes

- $\mathbb{Q}$, $\mathbb{Q}_n$ [Laflamme, Nguyen Van Thé, Sauer], $\mathbb{Q}_{\mathbb{Q}}$, $(\mathbb{Q}_{\mathbb{Q}})_n$,
- Rado graph, all structures in [LSV], generic k-partite graph, ordered versions of these.

Methodology for SDAP$^+$ Structures

1. Given enumerated $\mathbf{K}$, form the induced coding tree of 1-types.
2. Take a diagonal sub-coding tree.
3. Use forcing to prove a Halpern-Läuchli-style theorem on diagonal coding trees.

   This yields indivisibility for all arities. [Coulson–D.–Patel, Part I]
4. For structures with only unary and binary relations, do induction argument to get one color per diagonal antichain representing a finite structure. (no envelopes needed!)
5. Show the upper bounds in (4) are exact BRD. [Coulson–D.–Patel, Part II]

I. Big Ramsey Degree Characterizations and Methods

(c) Posets with Linear Extension and Carlson-Simpson's Ramsey Theorem for Parameter Words

The Generic Partial Order with Linear Extension

Let $\mathcal{P}$ be the Fraïssé class of finite partial orders with linear extensions. $\mathbf{P} = \mathrm{Flim}(\mathcal{P})$.

$\mathcal{L} = \{\leq, \prec\}$. For $\mathbf{A} \in \mathcal{P}$, $(v \leq w \wedge v \neq w) \Rightarrow v \prec w$.

Theorem (Hubička, 2020+)

The generic partial order with linear extension has finite big Ramsey degrees.

- Hubička also gave a short proof of finite BRD for the triangle-free Henson graph. Interestingly, this proof directly yields indivisibility.

The Generic Partial Order with Linear Extension

Theorem (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker, 2023+)

The generic partial order with linear extension has big Ramsey degrees characterized by poset diaries.

Words encoding partial orders

$\Sigma = \{\mathrm{L}, \mathrm{X}, \mathrm{R}\}$ is the alphabet, ordered by $\mathrm{L} <_{\mathrm{lex}} \mathrm{X} <_{\mathrm{lex}} \mathrm{R}$.

Σ^* is set of all finite words in the alphabet Σ. $\leq_{\mathrm{lex}}$ extends to Σ^*.

$$w = w_0 w_1 \dots w_{|w|-1}$$

Definition (Partial order $(\Sigma^*, \preceq)$)

For $w, w' \in \Sigma^*$, we set $w \prec w'$ if and only if there exists i such that:

1. $0 \leq i < \min(|w|, |w'|)$,
2. $(w_i, w'_i) = (\mathrm{L}, \mathrm{R})$,
3. $w_j \leq_{\mathrm{lex}} w'_j$ for every $0 \leq j < i$.

- $(\Sigma^*, \preceq)$ is a universal partial order and $(\Sigma^*, \leq_{\mathrm{lex}})$ is a linear extension of it. (Hubička, 2020+)

Carlson–Simpson Ramsey Theorem for Parameter Words

Let $\{\lambda_i : i < \omega\}$ be parameters.

For $n \leq \omega$, given an n-parameter word W and a parameter word s of length $k \leq n$, $W(s)$ is the word created by replacing each occurrence of λ_i, $i < k$, by s_i and truncating before first occurrence of λ_k in W.

Theorem (Carlson–Simpson, 1984)

If Σ^* is colored with finitely many colors, then there is an infinite-parameter word W such that $W[\Sigma^*] := \{W(s) : s \in \Sigma^*\}$ is monochromatic.

Method Remarks

- Apply Carlson–Simpson Theorem on a universal poset to get upper bounds. Afterward, pull out an enumerated copy of $\mathbb{P}$.
- Steps are similar to classic approach with Milliken's Theorem, BUT it can handle posets and $\mathcal{H}_3$ (but not $\mathcal{H}_4$).
- Forcing methods on coding trees fail for the generic partial order.

Poset Diaries

For $\ell > 0$ and words $w, w' \in \Sigma_\ell^*$, write $w \trianglelefteq w'$ iff $w_i \leq_{\text{lex}} w'_i$ for every $0 \leq i < \ell$. $w \perp w'$ iff w and w' are $\trianglelefteq$-incomparable.

$S \subseteq \Sigma^*$ is a **poset-diary** if S is a diagonal antichain in $(\Sigma^*, \sqsubseteq)$ and precisely one of the following four conditions is satisfied for every level ℓ with $0 \leq \ell < \sup_{w \in S} |w|$:

(1) Leaf.

(2) Splitting: One node splits into X,R.

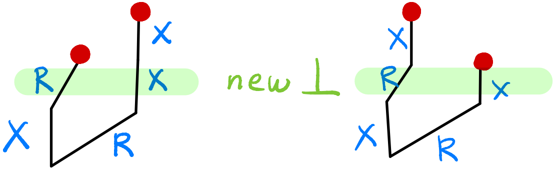
(3) New $\perp$.

(4) New relation $\prec$.

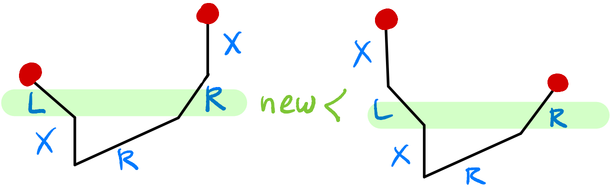
(3) and (4) are the 'interesting levels'.

Examples of Poset Diaries

Antichain of size 2



Chain of size 2



Hubička has calculated 464 poset diaries of partial orders of size 3.

I. Summary: BRD's and Diaries

All Diaries characterizing exact big Ramsey degrees (so far) involve

(a) Diagonal antichains

(b) passing types or interesting levels

Some (restricted FAP, posets) also involve

(c) essential age-changes/interesting levels

Some (restricted FAP) also involve

(d) controlled coding levels and paths.

Minimize ages, but make the changes happen as slowly as possible.

II. Infinite-dimensional Ramsey Theory on ω

Ramsey subsets of the Baire space

A subset $\mathcal{X}$ of $[\omega]^\omega$ is **Ramsey** if each for $M \in [\omega]^\omega$, there is an $N \in [M]^\omega$ such that $[N]^\omega \subseteq \mathcal{X}$ or $[N]^\omega \cap \mathcal{X} = \emptyset$.

Ramsey's Theorem (topological form). For any m and r, if $\mathcal{X} \subseteq [\omega]^\omega$ is a union of basic clopen sets of the form $[s, \omega]$ where $s \in [\omega]^m$, then $\mathcal{X}$ is Ramsey.

A subset $\mathcal{X}$ of $[\omega]^\omega$ is **Ramsey** if each for $M \in [\omega]^\omega$, there is an $N \in [M]^\omega$ such that $[N]^\omega \subseteq \mathcal{X}$ or $[N]^\omega \cap \mathcal{X} = \emptyset$.

A subset $\mathcal{X}$ of $[\omega]^\omega$ is **Ramsey** if each for $M \in [\omega]^\omega$, there is an $N \in [M]^\omega$ such that $[N]^\omega \subseteq \mathcal{X}$ or $[N]^\omega \cap \mathcal{X} = \emptyset$.

Ramsey's Theorem (topological form). For any m and r, if $\mathcal{X} \subseteq [\omega]^\omega$ is a union of basic clopen sets of the form $[s, \omega]$ where $s \in [\omega]^m$, then $\mathcal{X}$ is Ramsey.

A subset $\mathcal{X}$ of $[\omega]^\omega$ is **Ramsey** if each for $M \in [\omega]^\omega$, there is an $N \in [M]^\omega$ such that $[N]^\omega \subseteq \mathcal{X}$ or $[N]^\omega \cap \mathcal{X} = \emptyset$.

AC $\Rightarrow \exists \mathcal{X} \subseteq [\omega]^\omega$ which is not Ramsey.

Solution: restrict to 'definable' sets.

A subset $\mathcal{X}$ of $[\omega]^\omega$ is **Ramsey** if each for $M \in [\omega]^\omega$, there is an $N \in [M]^\omega$ such that $[N]^\omega \subseteq \mathcal{X}$ or $[N]^\omega \cap \mathcal{X} = \emptyset$.

Nash-Williams Thm. Clopen sets are Ramsey.

Galvin–Prikry Thm. Borel sets are Ramsey.

Silver Thm. Analytic sets are Ramsey.

Ellentuck Thm. A set is completely Ramsey iff it has the property of Baire in the Ellentuck topology.

Ellentuck Theorem

Ellentuck topology: refines the metric topology with basic open sets

$$[s, A] = \{B \in [\omega]^\omega : s \sqsubset B \subseteq A\}.$$

Ellentuck Theorem

Ellentuck topology: refines the metric topology with basic open sets

$$[s, A] = \{B \in [\omega]^\omega : s \sqsubset B \subseteq A\}.$$

Theorem (Ellentuck)

A set $\mathcal{X} \subseteq [\omega]^\omega$ satisfies

$$(*) \qquad \forall [s, A] \ \ \exists B \in [s, A] \text{ such that } [s, B] \subseteq \mathcal{X} \text{ or } [s, B] \cap \mathcal{X} = \emptyset$$

iff $\mathcal{X}$ has the property of Baire with respect to the Ellentuck topology.

Ellentuck Theorem

Ellentuck topology: refines the metric topology with basic open sets

$$[s, A] = \{B \in [\omega]^\omega : s \sqsubset B \subseteq A\}.$$

Theorem (Ellentuck)

A set $\mathcal{X} \subseteq [\omega]^\omega$ satisfies

$(*) \qquad \forall [s, A] \ \exists B \in [s, A]$ *such that* $[s, B] \subseteq \mathcal{X}$ *or* $[s, B] \cap \mathcal{X} = \emptyset$

iff $\mathcal{X}$ has the property of Baire with respect to the Ellentuck topology.

$(*)$ is called completely Ramsey by Galvin–Prikry and Ramsey by Todorcevic.

Ellentuck Theorem

Ellentuck topology: refines the metric topology with basic open sets

$$[s, A] = \{B \in [\omega]^\omega : s \sqsubset B \subseteq A\}.$$

Theorem (Ellentuck)

A set $\mathcal{X} \subseteq [\omega]^\omega$ satisfies

$$(*) \qquad \forall [s, A] \ \exists B \in [s, A] \text{ such that } [s, B] \subseteq \mathcal{X} \text{ or } [s, B] \cap \mathcal{X} = \emptyset$$

iff $\mathcal{X}$ has the property of Baire with respect to the Ellentuck topology.

$(*)$ is called completely Ramsey by Galvin–Prikry and Ramsey by Todorcevic.

The Ellentuck space is the prototype for **topological Ramsey spaces**:

Ellentuck Theorem

Ellentuck topology: refines the metric topology with basic open sets

$$[s, A] = \{B \in [\omega]^\omega : s \sqsubset B \subseteq A\}.$$

Theorem (Ellentuck)

A set $\mathcal{X} \subseteq [\omega]^\omega$ satisfies

$(*)$ $\quad \forall [s, A] \ \exists B \in [s, A]$ *such that* $[s, B] \subseteq \mathcal{X}$ *or* $[s, B] \cap \mathcal{X} = \emptyset$

iff $\mathcal{X}$ has the property of Baire with respect to the Ellentuck topology.

$(*)$ is called completely Ramsey by Galvin–Prikry and Ramsey by Todorcevic.

The Ellentuck space is the prototype for **topological Ramsey spaces**: Points are infinite sequences, topology is induced by finite heads and infinite tails, and every subset with the property of Baire satisfies $(*)$.

Examples of Topological Ramsey Spaces

- Ellentuck space
- Milliken strong trees
- Carlson-Simpson Parameter Words

For more on (topological) Ramsey spaces, see Todorcevic's 2010 book, *Introduction to Ramsey spaces*.

III. Infinite-dimensional Structural Ramsey Theory

KPT Question

Problem 11.2 in [KPT 2005]. Given a homogeneous structure $\mathbf{K}$, find the right notion of 'definable set' so that all definable subsets of $\binom{\mathbf{K}}{\mathbf{K}}$ are Ramsey.

KPT Question

Problem 11.2 in [KPT 2005]. Given a homogeneous structure $\mathbf{K}$, find the right notion of 'definable set' so that all definable subsets of $\binom{\mathbf{K}}{\mathbf{K}}$ are Ramsey.

We assume the universe of $\mathbf{K}$ is ω so that $\binom{\mathbf{K}}{\mathbf{K}}$ is a subspace of $[\omega]^\omega$.

KPT Question

Problem 11.2 in [KPT 2005]. Given a homogeneous structure $\mathbf{K}$, find the right notion of 'definable set' so that all definable subsets of $\binom{\mathbf{K}}{\mathbf{K}}$ are Ramsey.

We assume the universe of $\mathbf{K}$ is ω so that $\binom{\mathbf{K}}{\mathbf{K}}$ is a subspace of $[\omega]^\omega$.

Constraint: Big Ramsey degrees.

Must fix a big Ramsey structure and work on subcopies (embeddings) of it.

Infinite-Dimensional Ramsey Theory for SDAP$^+$ structures

Theorem (D. 2022+)

Let $\mathbf{K}$ *be a Fraïssé structure satisfying SDAP*$^+$ *with finitely many relations of arity at most two. Let* Δ *be a strong diary representing* $\mathbf{K}$*. Then every Borel subset of* $\mathcal{R}(\Delta)$ *is completely Ramsey.*

Examples: Rado graph, k-partite graphs, ordered versions.

Proof follows Galvin-Prikry but uses forcing for a stronger Pigeonhole.

Infinite-Dimensional Ramsey Theory for SDAP$^+$ structures

Theorem (D. 2022+)

Let $\mathbf{K}$ be a Fraïssé structure satisfying SDAP$^+$ with finitely many relations of arity at most two. Let Δ be a strong diary representing $\mathbf{K}$. Then every Borel subset of $\mathcal{R}(\Delta)$ is completely Ramsey.

Examples: Rado graph, k-partite graphs, ordered versions.

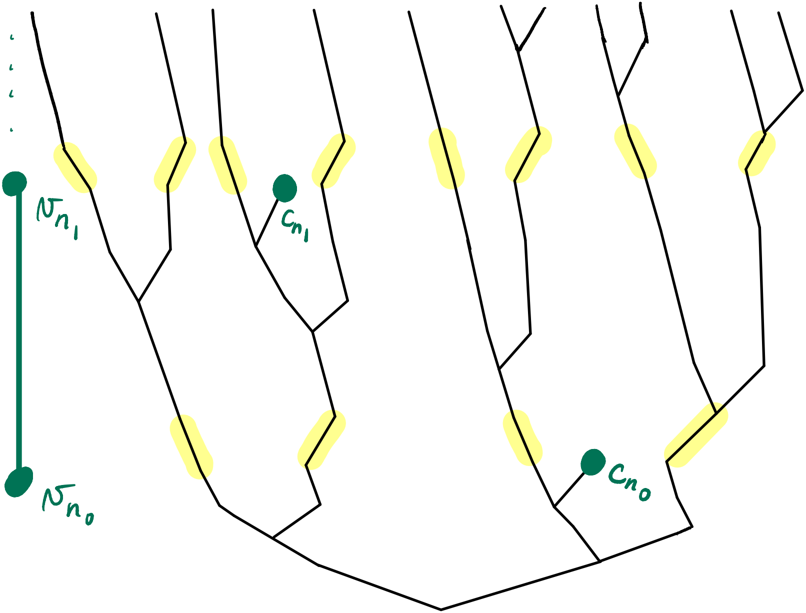
Proof follows Galvin-Prikry but uses forcing for a stronger Pigeonhole.

Corollary

If $\mathbf{K}$ has a certain amount of rigidity, Axiom A.3(2) of Todorcevic also holds, so we obtain analogues of Ellentuck's Theorem.

Examples: The rationals, $\mathbb{Q}_n$, $\mathbb{Q}_{\mathbb{Q}}$.

Diagonal Diary for the Rado Graph



We wanted to see if we could get a stronger ∞-dimensional theorem for the Rado graph and also for k-clique-free graphs.

Infinite-dimensional Ramsey Theory

Theorem (D.–Zucker)

Fix a finitely constrained binary free amalgamation class $\mathcal{K}$ and let $\mathbf{K} = Flim(\mathcal{K})$. Then $\mathbf{K}$ has infinite-dimensional Ramsey theory which directly recovers exact big Ramsey degrees in (BCDHKVZ 2021).

The strength of the theorem ranges from 'Souslin-measurable sets are Ramsey' (more than a Silver theorem analogue) to an analogue of the Ellentuck Theorem.

Abstract Ramsey Theorem

Theorem (Todorcevic)

Suppose that $(\mathcal{R}, \mathcal{S}, \le, \le_{\mathcal{R}})$ with finite restriction maps satisfying axioms **A.1**–*A.4, and that $\mathcal{S}$ is closed. Then the field of $\mathcal{S}$-Ramsey subsets of $\mathcal{R}$ is closed under the Souslin operation and it coincides with the field of $\mathcal{S}$-Baire subsets of $\mathcal{R}$.*

When $\mathcal{R} = \mathcal{S}$, this theorem implies the Abstract Ellentuck Theorem.

Theorem (D.–Zucker)

The conclusion of the above theorem still holds when axiom **A.3(2)** *is replaced by the weaker existence of an* **A.3(2)**-*ideal.*

IV. More Directions

- Non-forcing proofs.
- Higher arities.
- Infinite-dimensional structural Ramsey theory.
- Reverse Mathematics.
- Topological dynamics correspondence.
- When exactly does $\mathcal{K}$ having small Ramsey degrees imply $\text{Flim}(\mathcal{K})$ has finite big Ramsey degrees?
- What amalgamation or other properties of $\mathcal{K}$ correspond to the characterization of its big Ramsey degrees?

Expository References

Dobrinen, *Ramsey theory of homogeneous structures: current trends and open problems*. Proceedings of the International Congress of Mathematicians, 2022 (to appear). arXiv:2110.00655

Dobrinen, *Ramsey theory on infinite structures and the method of strong coding trees*. Contemporary logic and computing, 444–467, Landsc. Log, 1, Coll. Publ., (2020)

Dobrinen, *Forcing in Ramsey theory*, RIMS Kokyuroku 2042, (2017), 17–33. arXiv:1704.03898

Thank you very much!

$\mathcal{S}$-Baire and $\mathcal{S}$-Ramsey sets

For $X \in \mathcal{S}$ and a a finite approximation to some member of $\mathcal{R}$,

$$[a, X] = \{A \in \mathcal{R} : A \leq_{\mathcal{R}} X \text{ and } a \sqsubset A\}$$

A set $\mathcal{X} \subseteq \mathcal{R}$ is **$\mathcal{S}$-Baire** if for every non-empty basic open set $[a, X]$ there is an $a \sqsubseteq b \in \mathcal{AR}$ and $Y \leq X$ in $\mathcal{S}$ such that $[b, Y] \neq \emptyset$ and $[b, Y] \subseteq \mathcal{X}$ or $[b, Y] \subseteq \mathcal{X}^c$.

$\mathcal{S}$-Ramsey requires $b = a$ and $Y \in [\text{depth}_X(a), X]$.

Axioms for Ramsey Spaces

$(\mathcal{R}, \mathcal{S}, \leq, \leq_{\mathcal{R}})$ and finite restrictions maps;
$\leq \subseteq \mathcal{S} \times \mathcal{S}$ and $\leq_{\mathcal{R}} \subseteq \mathcal{R} \times \mathcal{S}$.

A.1 (Sequencing) For any choice of $\mathcal{P} \in \{\mathcal{R}, \mathcal{S}\}$,

(1) $M|_0 = N|_0$ for all $M, N \in \mathcal{P}$,
(2) $M \neq N$ implies that $M|_n \neq N|_n$ for some n,
(3) $M|_m = N|_n$ implies $m = n$ and $M|_k = N|_k$ for all $k \leq m$.

A.2 (Finitization) There is a transitive, reflexive relation $\leq_{\mathrm{fin}} \subseteq \mathcal{AS} \times \mathcal{AS}$ and a relation $\leq_{\mathrm{fin}}^{\mathcal{R}} \subseteq \mathcal{AR} \times \mathcal{AR}$ which are finitizations of the relations $\leq$ and $\leq_{\mathcal{R}}$, meaning that the following hold:

(1) $\{a : a \leq_{\mathrm{fin}}^{\mathcal{R}} x\}$ and $\{y : y \leq_{\mathrm{fin}} x\}$ are finite for all $x \in \mathcal{S}$,
(2) $X \leq Y$ iff $\forall m\, \exists n$ such that $X|_m \leq_{\mathrm{fin}} Y|_n$,
(3) $A \leq_{\mathcal{R}} X$ iff $\forall m\, \exists n$ such that $A|_m \leq_{\mathrm{fin}}^{\mathcal{R}} X|_n$,
(4) $\forall a \in \mathcal{AR}\ \forall x, y \in \mathcal{AS}\ [a \leq_{\mathrm{fin}}^{\mathcal{R}} x \leq_{\mathrm{fin}} y \ \rightarrow \ a \leq_{\mathrm{fin}}^{\mathcal{R}} y]$,
(5) $\forall a, b \in \mathcal{AR}\ \forall x \in \mathcal{AS}\ [a \sqsubseteq b$ and $b \leq_{\mathrm{fin}}^{\mathcal{R}} x \ \rightarrow \ \exists y \sqsubseteq x\ a \leq_{\mathrm{fin}}^{\mathcal{R}} y]$.

Todorcevic's Axioms 3 and 4 for Ramsey Spaces

A.3 (Amalgamation)

(1) $\forall a \in \mathcal{AR}\ \forall Y \in \mathcal{S}$,

$$[d = \mathrm{depth}_Y(a) < \infty \ \rightarrow\ \forall X \in [d, Y]\ ([a, X] \neq \emptyset)],$$

(2) $\forall a \in \mathcal{AR}\ \forall X, Y \in \mathcal{S}$, letting $d = \mathrm{depth}_Y(a)$,

$$[X \leq Y \text{ and } [a, X] \neq \emptyset \ \rightarrow\ \exists Y' \in [d, Y]\ ([a, Y'] \subseteq [a, X])].$$

A.4 (Pigeonhole) Suppose $a \in \mathcal{AR}_k$ and $\mathcal{O} \subseteq \mathcal{AR}_{k+1}$. Then for every $Y \in \mathcal{S}$ such that $[a, Y] \neq \emptyset$, there exists $X \in [Y|_d, Y]$, where $d = \mathrm{depth}_Y(a)$, such that the set $\{A|_{k+1} : A \in [a, X]\}$ is either contained in $\mathcal{O}$ or is disjoint from $\mathcal{O}$.

A.3(2)-ideals

An ideal $\mathcal{I} \subseteq \mathcal{S} \times \mathcal{S}$ is a set satisfying

- $(X, Y) \in \mathcal{I} \Rightarrow X \leq Y$.
- $(X, Y) \in \mathcal{I}$ and $Z \leq X \Rightarrow (Z, Y) \in \mathcal{I}$.

$\mathcal{I}$ is an **A.3(2)-ideal** if additionally

- $\forall Y \in \mathcal{S}\ \forall n < \omega\ \exists Y' \in \mathcal{S}$ with $(Y', Y) \in \mathcal{I}$ and $Y'|_n = Y|_n$.
- If $(X, Y) \in \mathcal{I}$ and $a \in \mathcal{AR}^{\mathcal{X}}$, there is $Y' \in \mathcal{S}$ with $Y' \in [\mathrm{depth_Y(a), Y}]$, $(Y', Y) \in \mathcal{I}$, and $[a, Y'] \subseteq [a, X]$.