Ramsey Theory on Infinite Structures, Part II

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Day 2: Big Ramsey Degree Methods and Characterizations

I. Fraïssé Theory and Big Ramsey Degrees

II. Milliken Methodology

   (a) Works for structures with universals that can be encoded as regularly finitely branching trees (unrestricted FAP).
   (b) Does not work for triangle-free Henson graph or FAP in general.

III. Coding trees of 1-types and 3 elements of BRD's

   (a) Enumerated structures and their coding trees of 1-types
   (b) Diagonal Antichains
   (c) Passing Types

IV. Forcing Ramsey Theorems on Coding Trees

   (a) Rado graph
   (b) Triangle-free Henson graph

V. Big Ramsey Degrees of Posets

I. Fraïssé Theory and Big Ramsey Degrees

I(a). Fraïssé Theory

Language $\mathcal{L}$: countably (for us, usually finitely) many relation symbols $\{R_i : i < n\}$, with k_i denoting the arity of R_i.

An $\mathcal{L}$-structure is an object $\mathbf{A} = \langle \mathrm{A}, R_0^{\mathbf{A}}, \dots, R_{n-1}^{\mathbf{A}} \rangle$, where A, the *universe* of $\mathbf{A}$, is non-empty and $R_i^{\mathbf{A}} \subseteq \mathcal{A}^{k_i}$.

For $\mathcal{L}$-structures $\mathbf{A}$ and $\mathbf{B}$, an embedding $e : \mathbf{A} \to \mathbf{B}$ is an injection on their universes $e : \mathrm{A} \to \mathrm{B}$ with the property that for all $i < n$,

$$R_i^{\mathbf{A}}(a_1, \dots, a_{n_i}) \longleftrightarrow R_i^{\mathbf{B}}(e(a_1), \dots, e(a_{n_i}))$$

The notions of copy of $\mathbf{A}$ in $\mathbf{B}$, substructure of $\mathbf{B}$, and isomorphism are natural.

$\mathbf{A} \leq \mathbf{B}$ means $\mathbf{A}$ embeds into $\mathbf{B}$.

$\mathbf{A} \cong \mathbf{B}$ means $\mathbf{A}$ is isomorphic to $\mathbf{B}$.

A class $\mathcal{K}$ of finite structures is called a **Fraïssé class** if it is nonempty, closed under isomorphisms, and satisfies

- Hereditary Property: Whenever $\mathbf{B} \in \mathcal{K}$ and $\mathbf{A}$ is a substructure of $\mathbf{B}$, then also $\mathbf{A} \in \mathcal{K}$ (for relational languages).
- Joint Embedding Property: For any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, there is a $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{A} \le \mathbf{C}$ and $\mathbf{B} \le \mathbf{C}$.
- Amalgamation Property: For any embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, there is a $\mathbf{D} \in \mathcal{K}$ and there are embeddings $r : \mathbf{B} \to \mathbf{D}$ and $s : \mathbf{C} \to \mathbf{D}$ such that $r \circ f = s \circ g$.

Note: For a relational language with finitely many relation symbols of any fixed arity, there are only countably many finite structures up to isomorphism.

Fraïssé Theory

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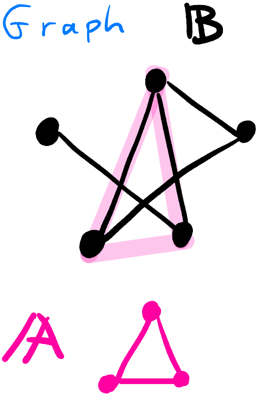
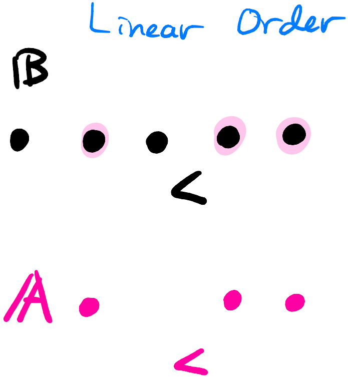
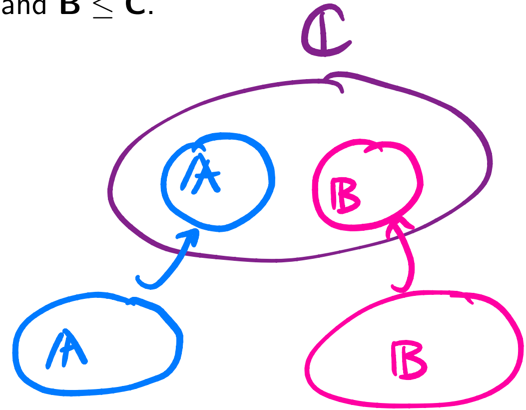
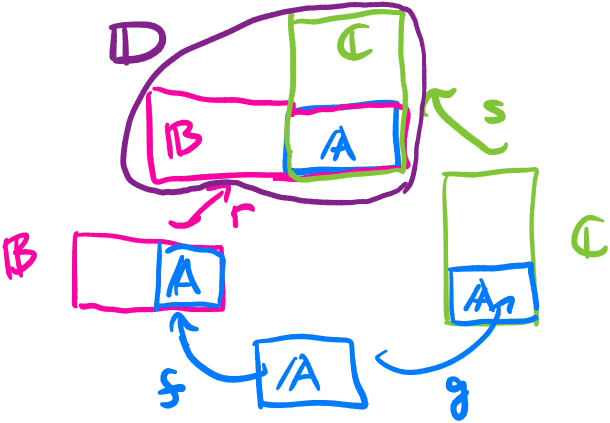




Joint Embedding Property: For any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, there is a $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{A} \leq \mathbf{C}$ and $\mathbf{B} \leq \mathbf{C}$.



Amalgamation Property: For any embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$, there is a $\mathbf{D} \in \mathcal{K}$ and there are embeddings $r : \mathbf{B} \to \mathbf{D}$ and $s : \mathbf{C} \to \mathbf{D}$ such that $r \circ f = s \circ g$.



Fraïssé Theory

Let $\mathcal{K}$ be a Fraïssé class of finite structures.

A structure $\mathbf{S}$ is **universal** for $\mathcal{K}$ if each structure in $\mathcal{K}$ embeds into $\mathbf{S}$.

An infinite structure $\mathbf{S}$ is **homogeneous** if each isomorphism between two finite substructures extends to an automorphism of $\mathbf{S}$.

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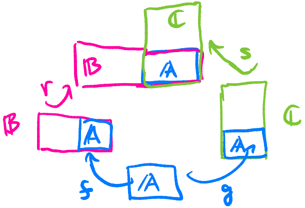
The **Fraïssé limit**

$$\mathbf{K} = \mathrm{Flim}(\mathcal{K})$$

is the unique (up to isomorphism) countable structure which is homogeneous and universal for $\mathcal{K}$.

A class $\mathcal{K}$ of finite structures satisfies the **disjoint amalgamation property (DAP)** (disjoint = strong (SAP)) if

given $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there is some $\mathbf{D} \in \mathcal{K}$ and embeddings $r : \mathbf{B} \to \mathbf{D}$ and $s : \mathbf{C} \to \mathbf{D}$ such that $r \circ f = s \circ g$, and $r[B] \cap s[C] = r \circ f[A] = s \circ g[A]$.

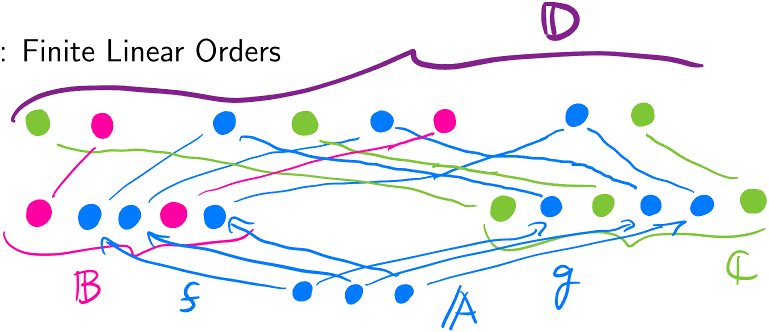


$\mathcal{K}$ satisfies the **free amalgamation property** (FAP) if it satisfies the DAP and moreover, $\mathbf{D}$ can be chosen so that $\mathbf{D}$ has no additional relations other than those inherited from $\mathbf{B}$ and $\mathbf{C}$.

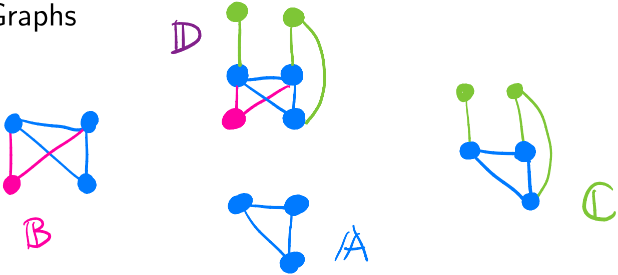
Disjoint and Free Amalgamation

Example 1: Finite Linear Orders

one ex. of a Disjt. Amalg.

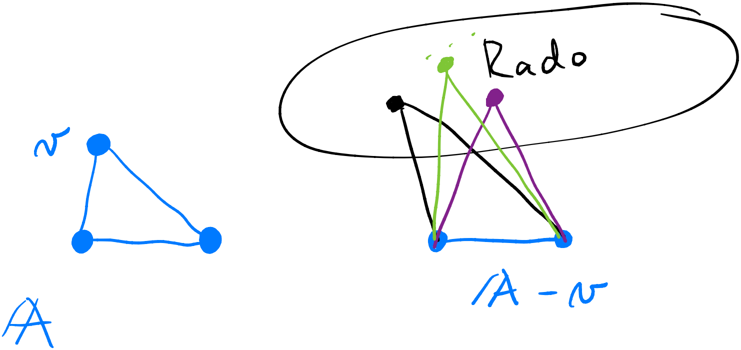


Example 2: Finite Graphs



Disjoint and Free Amalgamation

DAP is equivalent to the **strong embedding property**: For any $\mathbf{A} \in \mathcal{K}$, $v \in \mathrm{A}$, and embedding $e : (\mathbf{A} - v) \to \mathbf{K}$, there are infinitely many different extensions of e to embeddings of $\mathbf{A}$ into $\mathbf{K}$.



This makes DAP classes good for Ramsey Theory.

I(b). Big Ramsey Degrees

Finite Structural Ramsey Theory

For structures $\mathbf{A}, \mathbf{B}$, write $\mathbf{A} \le \mathbf{B}$ iff $\mathbf{A}$ embeds into $\mathbf{B}$.

$\binom{\mathbf{B}}{\mathbf{A}}$ denotes the set of all copies of $\mathbf{A}$ in $\mathbf{B}$.

A class $\mathcal{K}$ of finite structures has the **Ramsey Property** if given $\mathbf{A} \le \mathbf{B}$ in $\mathcal{K}$ and r, there is $\mathbf{C} \in \mathcal{K}$ so that

$$\forall \chi : \binom{\mathbf{C}}{\mathbf{A}} \to r \quad \exists \mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}, \ \chi \restriction \binom{\mathbf{B}'}{\mathbf{A}} \text{ is constant.}$$

Lots of work done! (e.g., Nešetřil–Rödl(77/83), Hubička–Nešetřil(2019))

Examples: The classes of **finite** linear orders, ordered graphs, ordered k-clique-free graphs, ordered k-regular hypergraphs, partial orders with linear extension,...

Small Remark.

Take the orders away and you get small Ramsey degrees.

A class $\mathcal{K}$ of finite structures has **small Ramsey degrees** if for each $\mathbf{A} \in \mathcal{K}$ there is a positive integer $t(\mathbf{A})$ such that for any $\mathbf{B} \in \mathcal{K}$ with $\mathbf{A} \leq \mathbf{B}$, there is a $\mathbf{C} \in \mathcal{K}$ with $\mathbf{B} \leq \mathbf{C}$ so that

$$\forall \chi : \binom{\mathbf{C}}{\mathbf{A}} \to r \quad \exists \mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}},\ \chi \restriction \binom{\mathbf{B}'}{\mathbf{A}}_{r,t(\mathbf{A})}$$

That is, for any coloring of the copies of $\mathbf{A}$ in $\mathbf{C}$ into r colors, there is a copy of $\mathbf{B}$ in $\mathbf{C}$ in which the copies of $\mathbf{A}$ take no more than $t(\mathbf{A})$ colors.

Topological Dynamics and Finite Structural RT

Theorem (Kechris–Pestov–Todorcevic, 2005)

A Fraïssé class $\mathcal{K}$ of finite structures has the Ramsey property if and only if $Aut(\mathbf{K})$ is extremely amenable, where $\mathbf{K}$ is the homogeneous structure universal for $\mathcal{K}$.

Let **K** be an infinite structure.

K has **finite big Ramsey degrees** if for each finite $\mathbf{A} \leq \mathbf{K}$, $\exists T$ such that $\forall r$, $\forall\ \chi : \binom{\mathbf{K}}{\mathbf{A}} \to r$, $\exists \mathbf{K}' \in \binom{\mathbf{K}}{\mathbf{K}}$ such that $|\chi \restriction \binom{\mathbf{K}'}{\mathbf{A}}| \leq T$.

The **big Ramsey degree** of **A** in **K**, $T(\mathbf{A})$, is the least such T.

We already saw that Devlin computed the big Ramsey degrees in the rationals.

Part of Question 11.2 of Kechris–Pestov–Todorcevic

Develop topological dynamics related to structural Ramsey theory for the

both finite and infinite dimensional

(i) The rationals;

(ii) The ordered Rado graph;

(iii) The k-clique-free ordered Henson graphs;

(iv) The random $\mathcal{A}$-free ordered hypergraph, where $\mathcal{A}$ is a set of finite irreducible ordered structures;

(v) The ordered rational Urysohn space;

(vi) The $\aleph_0$-dimensional vector space over a finite field with the canonical ordering;

(vii) The countable atomless Boolean algebra with the canoncial ordering.

Theorem (Zucker, 2019)

If $\mathbf{K}$ *has a big Ramsey structure, then* $Aut(\mathbf{K})$ *admits a unique universal completion flow.*

A **big Ramsey structure** for a Fraïssé structure $\mathbf{K}$ is an optimal (minimal) expansion $\mathbf{K}^*$ which produces exact big Ramsey degrees in a way that coheres.

A big Ramsey structure for $\mathbb{Q}$ is an expansion that encodes a diagonal antichain representing $\mathbb{Q}$.

Big Ramsey Degrees are almost always > 1

Let $\mathcal{K}$ be a Fraïssé class with limit $\mathbf{K}$.

Except for vertex colorings, exact analogues of Ramsey's Theorem usually fail.

- If $|\mathrm{Aut}(\mathbf{K})| > 1$, then $\exists \mathbf{A} \in \mathcal{K}$ with $T(\mathbf{A}) > 1$, or infinite.
  (Hjorth 2008)

Big Ramsey Degree results, a sampling

- 1933. $T(\text{Pairs}, \mathbb{Q}) \geq 2$. (Sierpiński)
- 1975. $T(\text{Edge}, \mathcal{R}) \geq 2$. (Erdős, Hajnal, Pósa)
- 1979. $(\mathbb{Q}, <)$: All BRD computed. (D. Devlin)
- 1986. $T(\text{Vertex}, \mathcal{H}_3) = 1$. (Komjáth, Rödl)
- 1989. $T(\text{Vertex}, \mathcal{H}_n) = 1$. (El-Zahar, Sauer)
- 1996. $T(\text{Edge}, \mathcal{R}) = 2$. (Pouzet, Sauer)
- 1998. $T(\text{Edge}, \mathcal{H}_3) = 2$. (Sauer)
- 2006, 2008. The Rado graph: All BRD characterized; computed. (Laflamme, Sauer, Vuksanović); (J. Larson)
- 2008, Rational Urysohn sphere. (Nguyen Van Thé)
- 2010. Dense Local Order $\mathbf{S}(2)$ and $\mathbb{Q}_n$: All BRD computed. (Laflamme, Nguyen Van Thé, Sauer)

Developments via coding trees and forcing (arxiv dates)

- 2017. Triangle-free Henson graphs: Very good Bounds. Exact bounds via small tweak in 2020. (D.)
- 2019. k-clique-free Henson graphs: Upper Bounds. (D.)
- 2020. Finitely constrained binary FAP: Upper Bounds. (Zucker)
- 2020. Exact BRD for binary (Part II) and indivisibility for higher arity (Part I) SDAP$^+$ structures. (Coulson, D., Patel)
- 2021. Binary rel. Forb($\mathcal{F}$): Exact BRD. (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker)
- Also some ∞-dimensional Ramsey theorems (tomorrow).

Developments not using forcing (arxiv dates)

- 2018. Certain homogeneous metric spaces: Upper Bounds. (Mašulović) category theory.
- 2019. 3-uniform hypergraphs: Upper Bounds. (Balko, Chodounský, Hubička, Konečný, Vena) Milliken Theorem.
- 2020. Circular directed graphs: Exact BRD Computed. (Dasilva Barbosa) category theory.
- 2020. Homogeneous partial order: Upper Bounds. (Hubička) Ramsey space of parameter words. First non-forcing proof for $\mathcal{H}_3$.
- 2021. Homogenous graphs with forbidden cycles (metric spaces): Upper Bounds. (Balko, Chodounský, Hubička, Konečný, Nešetřil, Vena) parameter words.
- 2023. Homogeneous partial order: Exact BRD. (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker) parameter words.
- 2023+. Certain Forb($\mathcal{F}$) binary and higher arities. (BCDHKNVZ) New methods.
- And more...

II. Classic (and current) methodology using Milliken's Theorem

Yesterday we saw that the big Ramsey degree of an m-sized subset of $\mathbb{Q}$ is exactly the number of diagonal antichains of size m.

We used Milliken's Theorem to obtain upper bounds, then made a diagonal antichain inside $2^{<\omega}$ representing a dense linear order, and we finished with a lower bound argument.

II. Classic Methodology using Milliken's Theorem

Theorem (Milliken, 1979)

Let T be a finitely branching subtree of $\omega^{<\omega}$ with no terminal nodes. Given $n \geq 1$ and a coloring of all n-strong subtrees of T into finitely many colors, there is an infinite strong subtree of T in which all n-strong subtrees have the same color.

For $\mathbf{K}$ = the Rationals, Rado graph, and more generally, FAP classes with finitely many binary relations and no forbidden substructures of size ≥ 3, one can

1. represent a universal structure via all nodes in $k^{<\omega}$
2. apply Milliken's envelopes to diagonal antichains
3. prove upper bounds exist.
4. Make a lower bound argument.

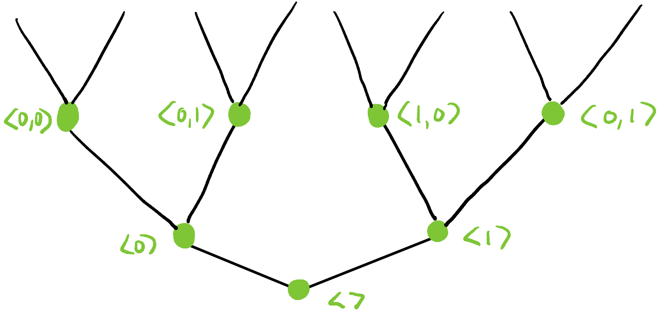
Rado graph and a universal

$2^{<\omega}$ with passing number 1 representing Edge and 0 representing Non-edge is a universal graph.

$\langle 1,0\rangle \; E \; \langle\rangle$

$\langle 1,0\rangle \; \not E \; \langle 1\rangle$

nodes of same length have no edges.



Unrestricted higher arity FAP classes

Theorem (Balko, Chodounský, Hubička, Konečný, Vena, 2022)

The 3-uniform generic hypergraph has finite big Ramsey degrees.

Proof uses product tree Milliken Theorem.

Theorem (Braunfeld, Chodounský, de Rancourt, Hubička, Kawach, Konečný, 2023)

Given a countable relational language $\mathcal{L}$ with finitely many relations of every arity > 1, let $\mathcal{K}$ be the Fraïssé class of finite unrestricted $\mathcal{L}$-structures. The Fraïssé limit has finite big Ramsey dgrees.

Proof uses [Laver 1984] Ramsey Theorem for product of infinitely many trees. They also prove that if there are infinitely many relations with the same arity, then there is a finite structure with BRD $= \infty$.

II(b). Where Milliken's Theorem is not useful

Milliken's Theorem and accompanying classic (and current for higher arities) methods are useful for proving upper bounds for finite big Ramsey degrees for FAP classes $\mathcal{K}$ for which there is a universal structure for $\mathcal{K}$ which can be represented by a tree $k^{<\omega}$ for some fixed k, or by the product of some uniformly branching trees.

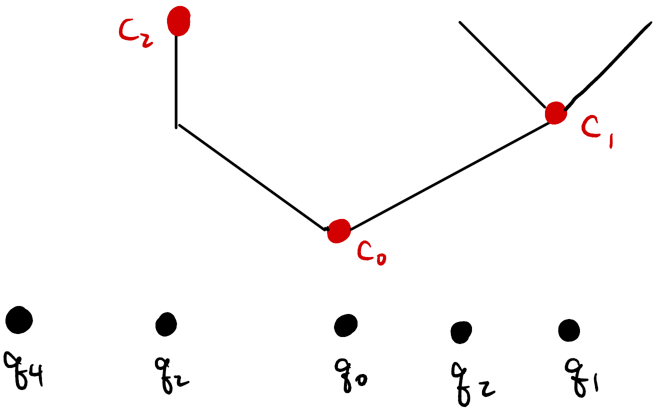
Milliken's Theorem cannot handle the triangle-free Henson graph, nor more generally, FAP classes for which some set of finite irreducible structures with universe larger than the arity of the largest relation in it are forbidden. e.g. triangle-free graphs.

III. Coding Trees of 1-types

III(a). Enumerated structures and their coding trees of 1-types

Coding tree for $\mathbb{Q}$

What happens next?



$\mathbb{Q}$ indexed by ω.

Big Ramsey Degrees for $\mathbb{Q}$ are characterized by diagonal antichains.

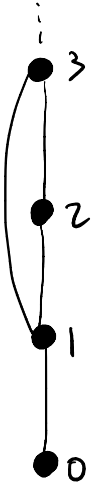
BRD of m-sized linear orders in $\mathbb{Q}$

= # of distinct types of diagonal antichains of size m

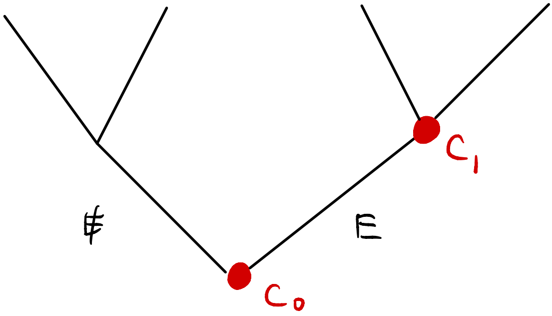
= # of diaries for m-sized subsets of $\mathbb{Q}$.

Coding tree for Rado graph

Rado with universe ω.



What happens next?



Coding tree for Rado graph

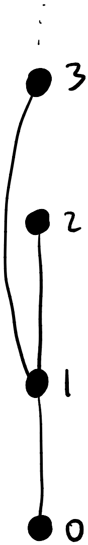
Big Ramsey Degrees for Rado graph are characterized by

1) Diagonal antichains
2) Passing Numbers

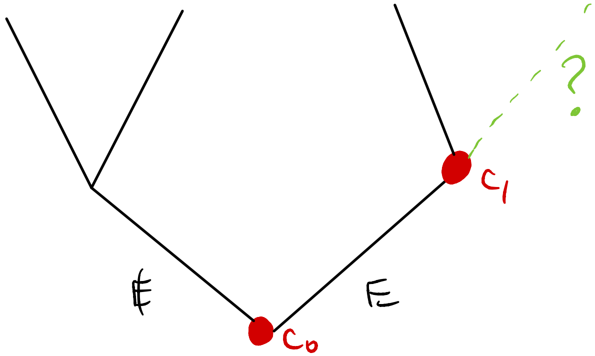
BRD of a finite graph G = # of distinct types of diagonal antichains encoding G.

Coding tree for triangle-free Henson graph

$\mathcal{H}_3$ with universe ω



What happens next?



Coding tree for triangle-free Henson graph

How are the BRD's of triangle-free graphs characterized?

1) Diagonal antichains

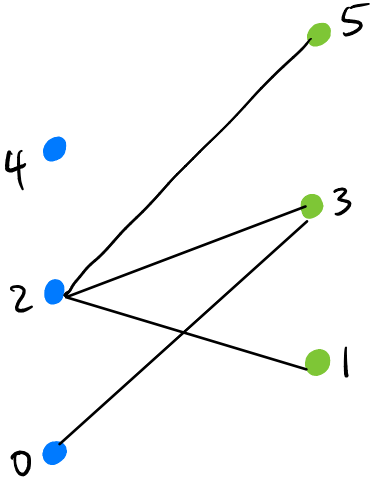
2) Passing Numbers

3) Something more. Can you guess?

Coding tree for homogeneous bipartite graph

Bipartite graph: $\mathcal{L} = \{U_0, U_1, E\}$

unary relations



Make the coding tree.

Coding tree for homogeneous bipartite graph

Big Ramsey degrees of bipartite graphs are characterized by

1) Diagonal antichains

2) Passing Numbers.

IV. Forcing Ramsey Theorems on Coding Trees.

Case Study: Triangle-Free Graphs

The **Henson graph**, $\mathcal{H}_3$, is the infinite homogeneous triangle-free graph into which every finite triangle-free graph embeds.

Previous Results:

- $T(\text{vertex}, \mathcal{H}_3) = 1$, Pigeonhole Principle (Komjáth–Rödl, 1986)
- $T(\text{Edge}, \mathcal{H}_3) = 2$ (Sauer, 1998)

Forcing opened new paths

The method of coding trees and using forcing on them was developed in December 2015 during my stay at the Newton Institute Semester on Set Theory. (but I initially started working on the problem in 2012)

- Start with the end in sight, and
- Try big machinery first: forcing.

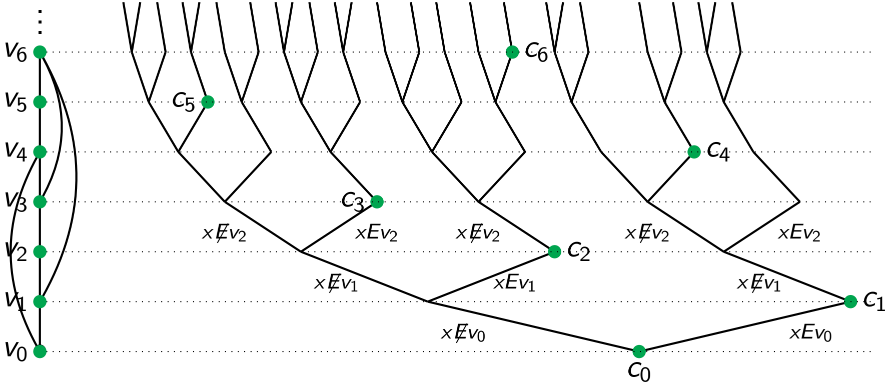
  precursor: Harrington's forcing proof of Halpern-Läuchli.

- Try to make a topological Ramsey space where each point is a Henson graph.

This last bullet would imply big Ramsey degrees and much more. This last part is recently completed in joint work with Andy Zucker

Share anecdotes. Friends help!

Coding Tree of 1-types for $\mathcal{H}_3$



Proof Outline

(1) Prove a version of Halpern-Läuchli for level sets in the coding tree.

(2) Do an inductive argument to prove a Milliken-like theorem for coding trees.

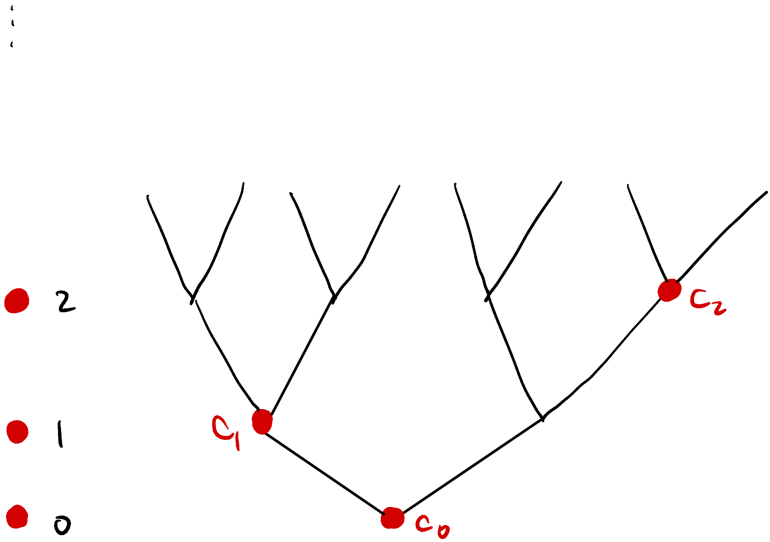
(3) Make a new notion of envelope.

(4) Figure out exactly what characterizes the BRD's.

(5) Show this characterization is exact (lower bounds argument).

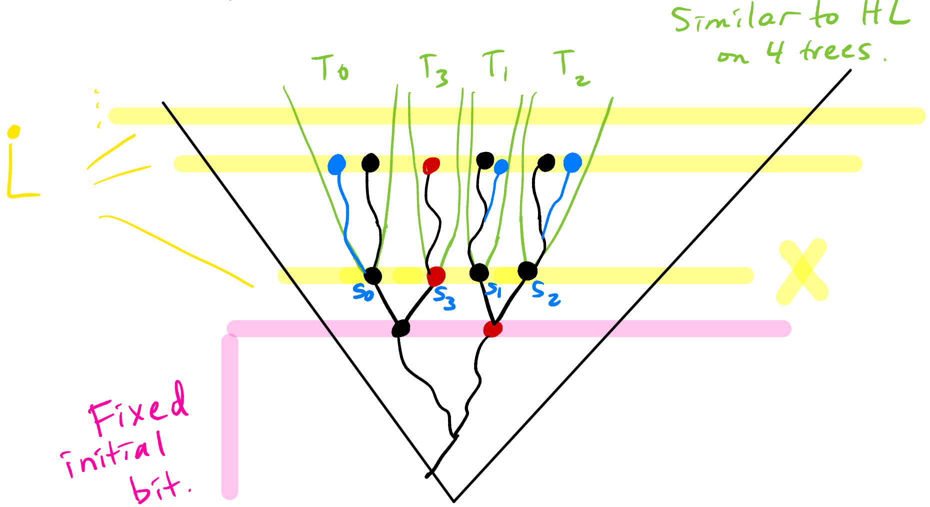
IV(a). Forcing Level Set Ramsey Theorems for the Rado graph

Rado graph



IV(a). Forcing in Rado graph coding tree (full branching version)

Given a coloring of the copies of X end-extending X.



IV(a). Forcing in Rado graph coding tree

Let $\kappa \longrightarrow (\aleph_1)^6_{\aleph_0}$.

$p \in \mathbb{P}$ iff $p : (d \times \vec{\delta}_p) \cup \{d\} \longrightarrow \bigcup_{i \le d} T_i \restriction \ell_p$

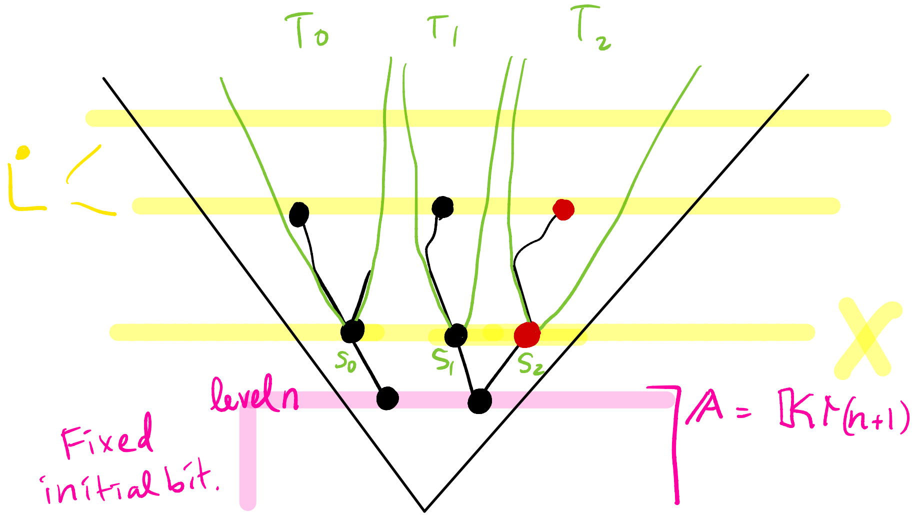
where $\vec{\delta}_p \in [\kappa]^{<\omega}$, $\ell_p \in L =$ levels with copies of X

$q \le p$ iff $q \supseteq p$ on $\mathrm{dom}(p)$.

$d = 3$ on previous slide.

IV(b). Forcing Level Set Ramsey Theorems for the triangle-free Henson graph

IV(b). Forcing triangle-free Henson graph coding tree



IV(b). Forcing triangle-free Henson graph coding tree

Let $\kappa \longrightarrow (\aleph_1)^6_{\aleph_0}$.

$p \in \mathbb{P}$ iff $p : (d \times \vec{\delta}_p) \cup \{d\} \longrightarrow \bigcup_{i \leq d} T_i \restriction \ell_p$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$, $\ell_p \in L$ = levels with copies of X

(in diag tree case, also require ran(p) can extend to structures as X does)

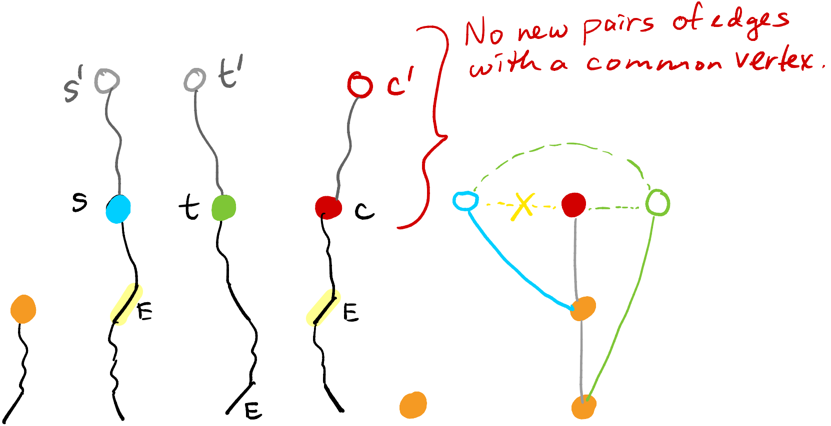
$q \leq p$ iff $q \supseteq p$ on dom(p)

and $A \cup \text{ran}(q \restriction \text{dom}(p))$ has same "age" as $A \cup \text{ran}(p)$.

IV(b). Forcing triangle-free Henson graph coding tree

Age{s, t, c} = {$\mathbb{B} \in \mathcal{G}_3$: $\mathbb{B}$ can be glued on top of the structure determined by c and below}

$q \le p$ requires $\mathrm{Age}(q \restriction \mathrm{dom}(p)) = \mathrm{Age}(p)$.

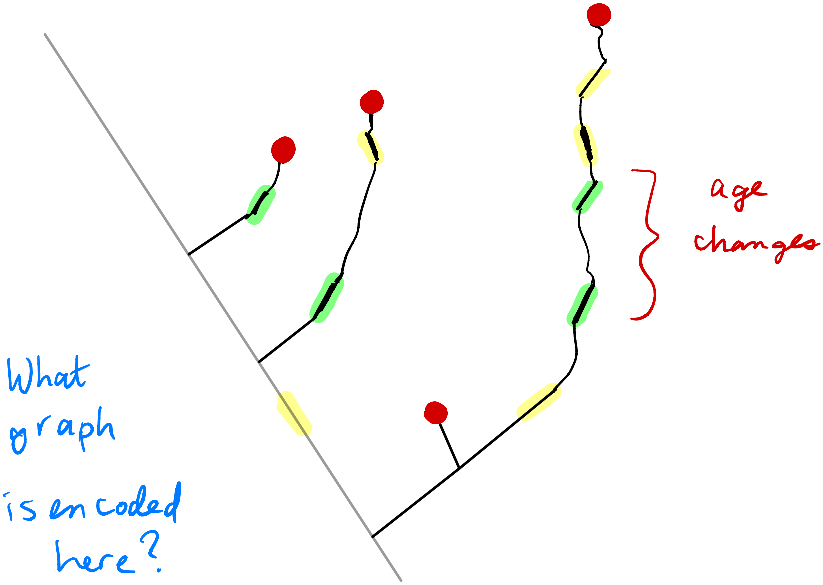


IV(b). Forcing triangle-free Henson graph coding tree

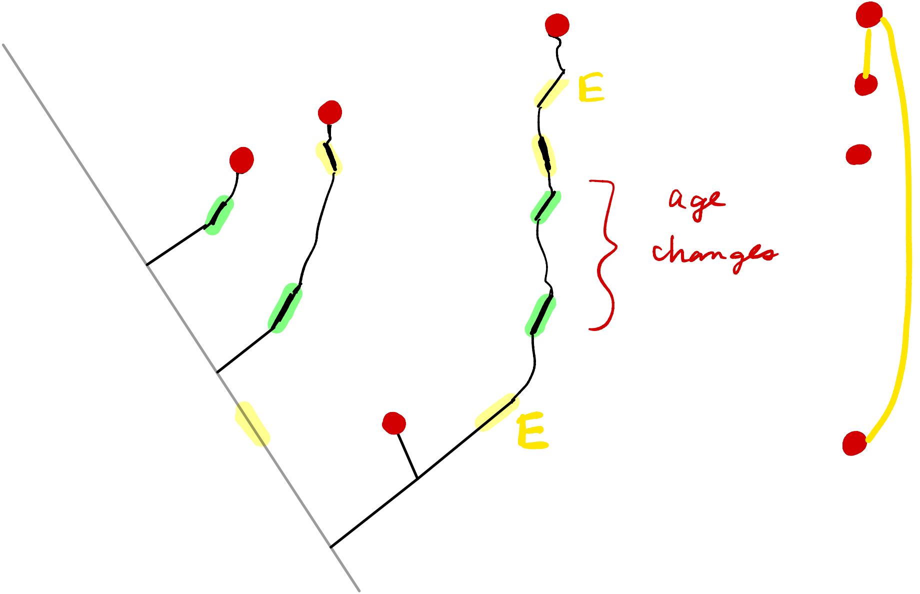
Big Ramsey degrees of triangle-free graphs are characterized by

1) Diagonal antichains
2) passing numbers
3) first levels off 0^ω (the leftmost branch)
4) least n where a pair of vertices has edges with v_n
5) A gadget at each coding node.

IV(b). Forcing triangle-free Henson graph coding tree



IV(b). Forcing triangle-free Henson graph coding tree



Upper bounds for Triangle-free Henson Graph

Theorem (D., JML 2020) and (JML 2023)

The triangle-free and more generally all k-clique-free Henson graphs have finite big Ramsey degrees.

Proofs directly reproduce indivisibility. because they work on diagonal subtrees of the coding tree of 1-types.

Exact BRD for triangle-free Henson graph

A small tweak of the trees in [D.2020] produces exact big Ramsey degrees.

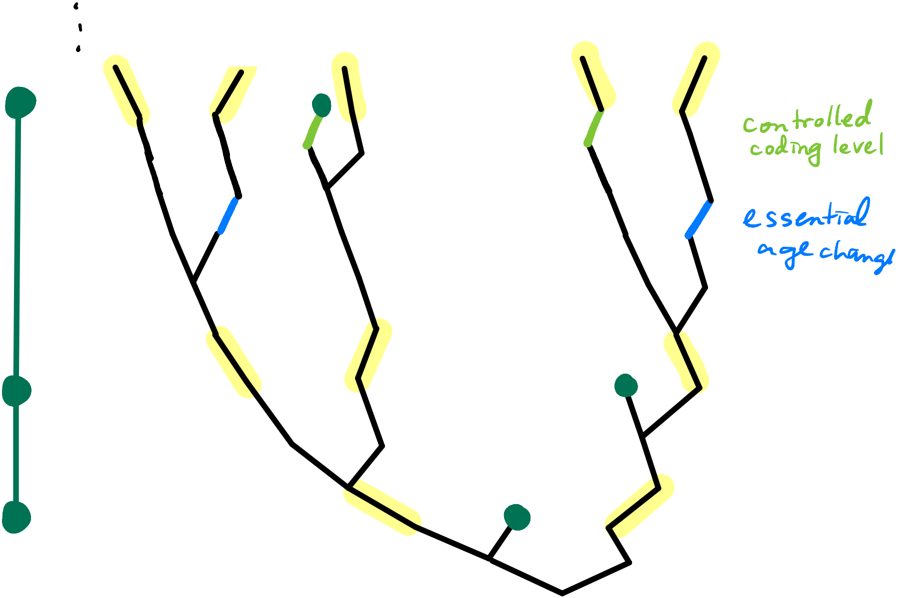
Theorem (D. and independently, Balko, Chodounský, Hubička, Konečný, Vena, Zucker, 2020)

Exact big Ramsey degrees of the triangle-free Henson graph are characterized.

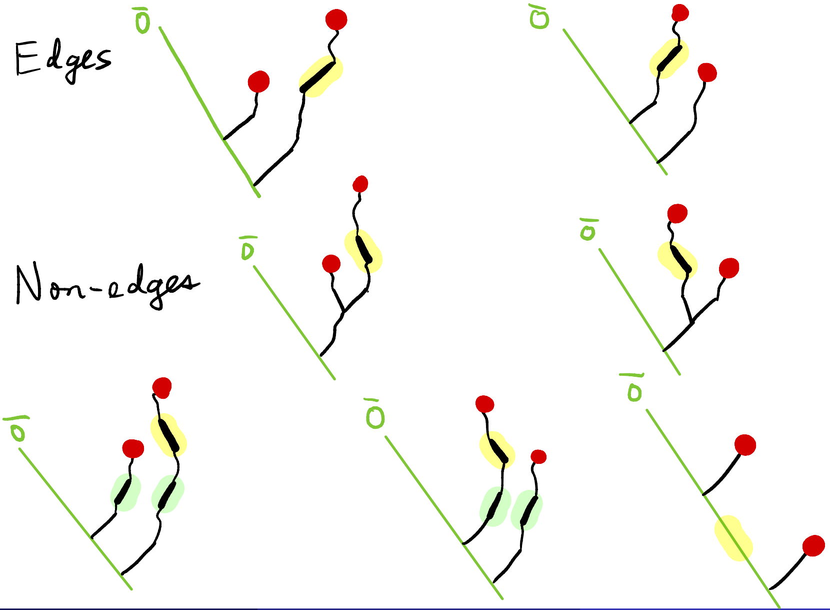
The characterization involves

(1) Diagonal antichains;
(2) Controlled age-change levels: first levels of pairs coding of edges with a common vertex in $\mathcal{H}_3$;
(3) Controlled coding levels;
(4) Controlled paths: first level off of leftmost branch.

A Strong (Diagonal) Diary for $\mathcal{H}_3$



BRD for pairs in Triangle-Free Henson Graph



Finitely Constrained Binary FAP Classes

Fix a language $\mathcal{L}$ with finitely many relations of arity at most 2.

An $\mathcal{L}$-structure is **irreducible** if any two vertices are in some relation: e.g., finite clique, finite tournament, triangle with 2 red edges and one blue edge.

Free amalgamation classes are exactly of the form Forb($\mathcal{F}$), where $\mathcal{F}$ is a set of finite **irreducible** structures.

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Free amalgamation classes are exactly of the form $\text{Forb}(\mathcal{F})$, where $\mathcal{F}$ is a set of finite **irreducible** structures.

Theorem (Zucker, 2022)

All finitely constrained binary FAP classes have finite big Ramsey degrees.

BRD of Finitely Constrained Binary FAP Classes

Theorem (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker, 2021+)

The exact big Ramsey degrees of finitely constrained binary FAP classes are characterized by the following:

1. *Diagonal antichains*
2. *Controlled splitting levels*
3. *Controlled age-change levels (essential changes in the class of structures which can be glued above a finite structure to make a member of $\mathcal{K}$)*
4. *Controlled coding levels (reducing the ages of the extending class as much as possible)*
5. *Controlled paths (only matter for non-trivial unary relations)*

Unexpected applications of coding trees and forcing to structures which behave like $\mathbb{Q}$ or the Rado graph:

II (b). Applications of Forcing and Coding Trees to SDAP^+ classes.

SDAP$^+$ $\Rightarrow$ Simple BRD

Theorem (Coulson–D.–Patel)

Let $\mathcal{L}$ be a finite relational language and let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit satisfying the Substructure Disjoint Amalgamation Property$^+$. Let $\mathbf{K} = Flim(\mathcal{K})$.

I. $\mathbf{K}$ is indivisible.

II. If $\mathcal{L}$ has no relations of arity greater than two, then $\mathbf{K}$ has big Ramsey degrees characterized by diagonal antichains.

This class of structures includes

- $\mathbb{Q}$, $\mathbb{Q}_n$ [Laflamme, Nguyen Van Thé, Sauer], $\mathbb{Q}_{\mathbb{Q}}$, $(\mathbb{Q}_{\mathbb{Q}})_n$,
- Rado graph, all structures in [LSV], generic k-partite graph, ordered versions of these.

Methodology for SDAP$^+$ Structures

1. Given enumerated **K**, form the induced coding tree of 1-types.
2. Take a diagonal sub-coding tree.
3. Use forcing to prove a Halpern-Läuchli-style theorem on diagonal coding trees.

   This yields indivisibility for all arities. [Coulson–D.–Patel, Part I]
4. For structures with only unary and binary relations, do induction argument to get one color per diagonal antichain representing a finite structure. (no envelopes needed!)
5. Show the upper bounds in (4) are exact BRD. [Coulson–D.–Patel, Part II]

V. The Homogeneous Poset with Linear Extension

V. The Generic Partial Order with Linear Extension

Let $\mathcal{P}$ be the Fraïssé class of finite partial orders with linear extensions. $\mathbf{P} = \text{Flim}(\mathcal{P})$.

$\mathcal{L} = \{\leq, \prec\}$. For $\mathbf{A} \in \mathcal{P}$, $(v \leq w \wedge v \neq w) \Rightarrow v \prec w$.

Theorem (Hubička, 2020+)

The generic partial order with linear extension has finite big Ramsey degrees.

- Hubička also gave a short proof of finite BRD for the triangle-free Henson graph. Interestingly, this proof directly yields indivisibility.

The Generic Partial Order with Linear Extension

Theorem (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker, 2023+)

The generic partial order with linear extension has big Ramsey degrees characterized by poset diaries.

Words encoding partial orders

$\Sigma = \{\mathrm{L}, \mathrm{X}, \mathrm{R}\}$ is the alphabet, ordered by $\mathrm{L} <_{\mathrm{lex}} \mathrm{X} <_{\mathrm{lex}} \mathrm{R}$.

Σ^* is set of all finite words in the alphabet Σ. $\leq_{\mathrm{lex}}$ extends to Σ^*.

$$w = w_0 w_1 \dots w_{|w|-1}$$

Definition (Partial order $(\Sigma^*, \preceq)$)

For $w, w' \in \Sigma^*$, we set $w \prec w'$ if and only if there exists i such that:

1. $0 \leq i < \min(|w|, |w'|)$,
2. $(w_i, w'_i) = (\mathrm{L}, \mathrm{R})$,
3. $w_j \leq_{\mathrm{lex}} w'_j$ for every $0 \leq j < i$.

- $(\Sigma^*, \preceq)$ is a universal partial order and $(\Sigma^*, \leq_{\mathrm{lex}})$ is a linear extension of it. (Hubička, 2020+)

Carlson–Simpson Ramsey Theorem for Parameter Words

Let $\{\lambda_i : i < \omega\}$ be parameters.

For $n \leq \omega$, given an n-parameter word W and a parameter word s of length $k \leq n$, $W(s)$ is the word created by replacing each occurrence of λ_i, $i < k$, by s_i and truncating before first occurrence of λ_k in W.

Theorem (Carlson–Simpson, 1984)

If Σ^* is colored with finitely many colors, then there is an infinite-parameter word W such that $W[\Sigma^*] := \{W(s) : s \in \Sigma^*\}$ is monochromatic.

Method Remarks

- Apply Carlson–Simpson Theorem on a universal poset to get upper bounds. Afterward, pull out an enumerated copy of $\mathbb{P}$.
- Steps are similar to classic approach with Milliken's Theorem, BUT it can handle posets and $\mathcal{H}_3$ (but not $\mathcal{H}_4$).
- Forcing methods on coding trees fail for the generic partial order.

Poset Diaries

For $\ell > 0$ and words $w, w' \in \Sigma_\ell^*$, write $w \trianglelefteq w'$ iff $w_i \leq_{\text{lex}} w'_i$ for every $0 \leq i < \ell$. $w \perp w'$ iff w and w' are $\trianglelefteq$-incomparable.

$S \subseteq \Sigma^*$ is a **poset-diary** if S is a diagonal antichain in $(\Sigma^*, \sqsubseteq)$ and precisely one of the following four conditions is satisfied for every level ℓ with $0 \leq \ell < \sup_{w \in S} |w|$:

(1) Leaf.

(2) Splitting: One node splits into X,R.

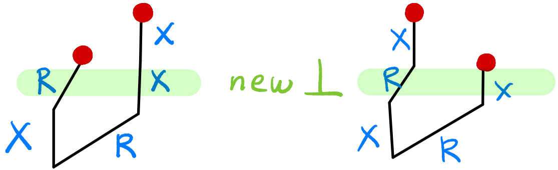
(3) New $\perp$.

(4) New relation $\prec$.

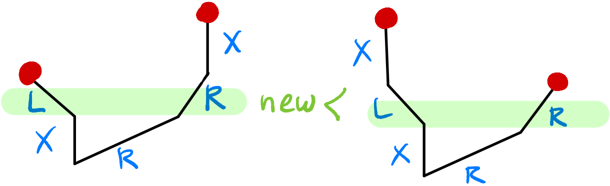
(3) and (4) are the 'interesting levels'.

Examples of Poset Diaries

Antichain of size 2



Chain of size 2



Hubička has calculated 464 poset diaries of partial orders of size 3.

I. Summary: BRD's and Diaries

All Diaries characterizing exact big Ramsey degrees (so far) involve

(a) Diagonal antichains

(b) passing types or interesting levels

Some (restricted FAP/posets) also involve

(c) essential age-changes/interesting levels

Some (restricted FAP) also involve

(d) controlled coding levels and paths.

Minimize ages, but make the changes happen as slowly as possible.

Next time,

∞-dimenstional Ramsey Theory!