Infinite-dimensional Ramsey theory on binary relational homogeneous structures

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Luminy Workshop on Set Theory

October 9–13, 2023

Research supported by NSF grant DMS-2300896

Ramsey's Theorem (finite-dimensional)

Theorem (Ramsey)

Given m, r and a coloring of $[\omega]^m$ into r colors, there is an $N \in [\omega]^\omega$ such that all members of $[N]^m$ have the same color.

A subset $\mathcal{X} \subseteq [\omega]^\omega$ is **Ramsey** if each for $M \in [\omega]^\omega$, there is an $N \in [M]^\omega$ such that $[N]^\omega \subseteq \mathcal{X}$ or $[N]^\omega \cap \mathcal{X} = \emptyset$.

Ramsey's Theorem (topological form). For any m and r, if $\mathcal{X} \subseteq [\omega]^\omega$ is a union of basic clopen sets of the form $[s, \omega]$ where $s \in [\omega]^m$, then $\mathcal{X}$ is Ramsey.

Infinite-dimensional Ramsey Theory

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Axiom of Choice $\Longrightarrow \exists \mathcal{X} \subseteq [\omega]^\omega$ which is not Ramsey.

Solution: restrict to 'definable' sets.

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Nash-Williams. Clopen sets are Ramsey.

Galvin. Open sets are Ramsey.

Galvin–Prikry. Borel sets are completely Ramsey.

Silver. Analytic sets are completely Ramsey.

Ellentuck. A set is completely Ramsey iff it has the property of Baire in the Ellentuck topology.

Louveau. Local version for tails in a Ramsey ultrafilter.

Ellentuck Theorem

Ellentuck topology: refines the metric topology with basic open sets

$$[s, A] = \{B \in [\omega]^\omega : s \sqsubset B \subseteq A\}.$$

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iff $\mathcal{X}$ has the property of Baire with respect to the Ellentuck topology.

$(*)$ is called completely Ramsey by Galvin–Prikry and Ramsey by Todorcevic.

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Topological Ramsey spaces: Points are infinite sequences, topology is induced by finite heads and infinite tails, and every subset with the property of Baire satisfies $(*)$.

(Carlson–Simpson 1990; Todorcevic 2010.)

Part of Question 11.2 of Kechris–Pestov–Todorcevic

Develop infinite-dimensional Ramsey theory for the

(i) Rationals;

(ii) Ordered Rado graph;

(iii) *k*-clique-free ordered Henson graphs;

(iv) Random $\mathcal{A}$-free ordered hypergraph, where $\mathcal{A}$ is a set of finite irreducible ordered structures;

(v) Ordered rational Urysohn space;

(vi) $\aleph_0$-dimensional vector space over a finite field with the canonical ordering;

(vii) The countable atomless Boolean algebra with the canoncial ordering.

A successful topological characterization should recover big Ramsey degrees exactly.

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(i) Rationals; D. 2022

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(iii) k-clique-free ordered Henson graphs; D.-Zucker 2023

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(v) Ordered rational Urysohn space;

(vi) $\aleph_0$-dimensional vector space over a finite field with the canonical ordering; Impossible for $\mathbb{F}_p$, $p \geq 3$. Nguyen Van Thé 2008

(vii) The countable atomless Boolean algebra with the canoncial ordering.

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(Infinite) Homogeneous Structures

A structure **K** is **homogeneous** if every isomorphism between two finite induced substructures of **K** extends to an automorphism of **K**.

Homogeneous structures are Fraïssé limits. Examples include the previous as well as the

- $(\mathcal{R}, E)$ Rado graph
- $(\mathcal{H}_k, E)$ k-clique-free Henson graphs, $k \geq 3$
- generic k-partite graph
- generic digraph
- random graph with Red and Blue edges omitting RRB and RBB triangles and Red 4-cliques
- generic partial order
- rationally ordered versions: $(\mathcal{R}, E, <)$, $(\mathcal{H}_k, E, <)$, ...
- Free superpositions of the above

Infinite-Dimensional Structural Ramsey Theory

$$\mathbf{K} \rightarrow^* (\mathbf{K})^{\mathbf{K}}$$

- Well-ordering **K** induces
  — a metric topology, like Baire space.
  — a tree of 1-types, which is preserved in any subcopy of **K**, inducing Big Ramsey Degrees (BRD).

Big Ramsey Degrees

Let $\mathcal{K}$ be a Fraïssé class with limit **K**.

K has **finite big Ramsey degrees** if for each finite $\mathbf{A} \le \mathbf{K}$, $\exists t$ such that $\forall r$, $\forall\ \chi : \binom{\mathbf{K}}{\mathbf{A}} \to r$, $\exists \mathbf{K}' \in \binom{\mathbf{K}}{\mathbf{K}}$ such that $|\chi \restriction \binom{\mathbf{K}'}{\mathbf{A}}| \le t$.

$$\mathbf{K} \to (\mathbf{K})^{\mathbf{A}}_{r,t}$$

The **big Ramsey degree** of **A** in $\mathbf{K} = \mathrm{BRD}(\mathbf{A}, \mathbf{K}) = \mathrm{BRD}(\mathbf{A})$ is the least such t.

- (Hjorth 2008) If $|\mathrm{Aut}(\mathbf{K})| > 1$, then $\mathcal{K}$ has some BRD > 1.

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BRD's are really about the optimal structural expansions for which Ramsey's Theorem holds. (canonical partitions) LSV

Zucker's notion of 'big Ramsey structure'.

Big Ramsey Degree results, a sampling

- 1933. BRD(Pairs, $\mathbb{Q}$) ≥ 2. (Sierpiński)
- 1975. BRD(Edge, $\mathcal{R}$) ≥ 2. (Erdős, Hajnal, Pósa)
- 1979. $(\mathbb{Q}, <)$: All BRD computed. (D. Devlin)
- 1986. BRD(Vertex, $\mathcal{H}_3$) $= 1$. (Komjáth, Rödl)
- 1989. BRD(Vertex, $\mathcal{H}_n$) $= 1$. (El-Zahar, Sauer)
- 1996. BRD(Edge, $\mathcal{R}$) $= 2$. (Pouzet, Sauer)
- 1998. BRD(Edge, $\mathcal{H}_3$) $= 2$. (Sauer)
- 2006, 2008. The Rado graph: All BRD characterized; computed. (Laflamme, Sauer, Vuksanović); (J. Larson)
- 2008. Ultrametric spaces with finite distance set: All BRD characterized. (Nguyen Van Thé)
- 2010. Dense Local Order $\mathbf{S}(2)$: All BRD computed. Also $\mathbb{Q}_n$. (Laflamme, Nguyen Van Thé, Sauer)

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∞ Structural RT via coding trees and forcing (arxiv dates)

- 2017. Triangle-free Henson graphs: FBRD foreshadowing ∞-diml Exact bounds via small tweak in 2020. (D.) and independently (BDHKVZ)
- 2019. k-clique-free Henson graphs: Upper Bounds. (D.)
- 2019. ∞-dimensional RT for Borel sets of Rado graphs. (D.)
- 2020. Binary rel. Forb($\mathcal{F}$): Upper Bounds. (Zucker)
- 2020. Exact BRD for binary SDAP$^+$ structures. (Coulson, D., Patel)
- 2021. Binary rel. Forb($\mathcal{F}$): Exact BRD. (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker)
- 2022. ∞-dimensional RT structures with SDAP$^+$. recovers Exact BRD. (D.)
- 2023+. ∞-dimensional RT for finitely constrained binary FAP. recovers Exact BRD. (D., Zucker)

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- 2023+. ∞-dimensional RT for finitely constrained binary FAP. recovers Exact BRD. (D., Zucker)

Developments not using forcing (arxiv dates)

- 2018. Certain homogeneous metric spaces: FBRD. (Mašulović) category th.
- 2019. 3-uniform hypergraphs: FBRD. (Balko, Chodounský, Hubička, Konečný, Vena) Milliken Theorem.
- 2020. Circular directed graphs: Exact BRD Computed. (Dasilva Barbosa) category theory.
- 2020. Homogeneous partial order: FBRD. (Hubička) Ramsey space of parameter words. First non-forcing proof for $\mathcal{H}_3$.
- 2021. Homogenous graphs with forbidden cycles (metric spaces): FBRD. (Balko, Chodounský, Hubička, Konečný, Nešetřil, Vena) param. words.
- 2023. Homogeneous partial order: Exact BRD. (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker) parameter words.
- 2023. Infinite languages, unrestricted structures: FBRD. (Braunfeld, Chodounský, de Rancourt, Hubička, Kawach, Konečný. Laver Theorem.
- 2023+. Many Forb($\mathcal{F}$), all arities, and more: FBRD. (BCDHKNVZ) New methods.
- 2023+. Pseudotrees. (Chodounský, D., Eskew, Weinert)

Abstract Ramsey Theorem (∞-diml Ramsey Theory)

Theorem (Todorcevic)

Suppose we are given a structure $(\mathcal{R}, \mathcal{S}, \leq, \leq_{\mathcal{R}})$ with finite restrictions maps satisfying Axioms A.1 to A.4, and that $\mathcal{S}$ is closed. Then the field of $\mathcal{S}$-Ramsey subsets of $\mathcal{R}$ is closed under the Souslin operation and it coincides with the field of $\mathcal{S}$-Baire subsets of $\mathcal{R}$.

$$\mathcal{R} = \mathcal{S} \implies \text{Abstract Ellentuck Theorem}$$

So if we could just show that our spaces of subcopies of **K** satisfy these four axioms, we'd be done.

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So if we could just show that our spaces of subcopies of **K** satisfy these four axioms, we'd be done. BUT

- BRDs preclude working with spaces of ALL subcopies of **K**.
- A.3(2) generally usually fails for Fraïssé structures.

Big Ramsey Degree Characterizations: Diaries

Big Ramsey degrees of a binary relational homogeneous structure **K** are characterized via enumerating the universe of **K** and forming the coding tree of 1-types and

I. Diagonal antichains (in the coding tree of 1-types);

II. Passing types;

III. Forbidden substructures also include

- I(a). Controlled splitting levels;
- II(a). Controlled coding triples;
- III(a). Maximal paths;
- III(b). Essential age-change levels (incremental changes in how much of a forbidden substructure is coded).

The Point

Any infinite-dimensional structural Ramsey theory must start by fixing a diary and then working with the space of all subcopies of that diary.

Infinite-Dimensional Ramsey Theory for SDAP$^+$ structures

Theorem (D.)

1. *Let* $\mathbf{K}$ *be a Fraïssé structure satisfying SDAP*$^+$ *with finitely many relations of arity at most two. Then for each (good) diary, the space of isomorphic subdiaries satisfies a Galvin-Prikry Theorem.*
2. *If* $\mathbf{K}$ *has a certain amount of rigidity, Axiom A.3(2) of Todorcevic also holds, so we obtain analogues of Ellentuck's Theorem.*

Proof Outline:

(1) "Force" a strengthened Pigeonhole Lemma for colorings of copies of a given level set.

(2) Prove that every Nash-Williams family restricts to a front or $\emptyset$ on some member of the space. uses 'combinatorial forcing'

(3) Use the PL to show that opens sets are CR* and that countable unions of CR* sets are CR*.

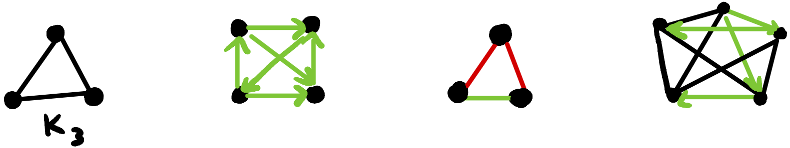
(4) Complements of CR* sets are CR*, hence Borel sets are CR*.

Finitely constrained binary relational FAP classes

A structure is **irreducible** if any two vertices are in some relation: e.g., finite clique, finite tournament, triangle with 2 red edges and one green edge.

Free amalgamation classes are exactly of the form Forb($\mathcal{F}$), where $\mathcal{F}$ is a set of finite irreducible structures.

Finitely constrained binary relational FAP classes



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∞-diml RT for binary finitely constrained FAP classes

Theorem (D., Zucker)

*Let **K** be a finitely constrained homogeneous structure with free amalgamation and finitely many relations of arity ≤ 2. Then **K** has an infinite-dimensional Ramsey theory which directly recovers the exact big Ramsey degrees in (BCDHKVZ 2021).*

Proof Outline:

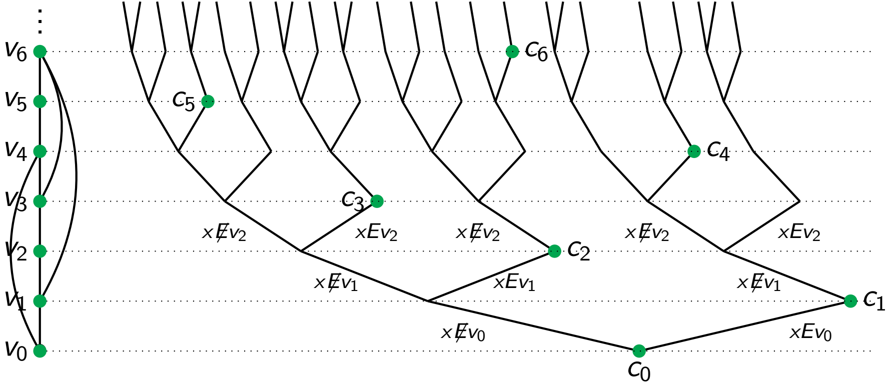
(1) Prove that a weaker version of A.3 suffices to guarantee the Abstract Ramsey Theorem.

(2) Show that certain two-sorted spaces of diaries satisfy weakened A.3(2).

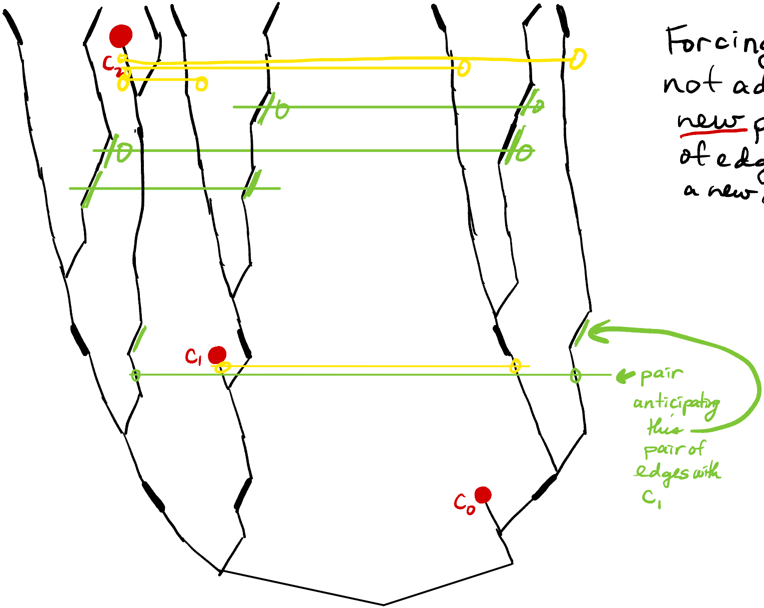
(3) "Force" a Pigeonhole Lemma for colorings of copies of a given level set.

Coding Tree of 1-types for $\mathcal{H}_3$

Enumerating the vertices of $\mathcal{H}_3$ induces the tree possibilities.



A Strong Diary Δ for $\mathcal{H}_3$



Forcing must not add new pairs of edges with a new vertex.

$\mathcal{S}$-Baire and $\mathcal{S}$-Ramsey sets

For $X \in \mathcal{S}$ and a finite approximation a to some member of $\mathcal{R}$,

$$[a, X] = \{A \in \mathcal{R} : A \leq_{\mathcal{R}} X \text{ and } a \sqsubset A\}$$

A set $\mathcal{X} \subseteq \mathcal{R}$ is **$\mathcal{S}$-Baire** if for every non-empty basic open set $[a, X]$ there is an $a \sqsubseteq b \in \mathcal{AR}$ and $Y \leq X$ in $\mathcal{S}$ such that $[b, Y] \neq \emptyset$ and $[b, Y] \subseteq \mathcal{X}$ or $[b, Y] \subseteq \mathcal{X}^c$.

$\mathcal{S}$-Ramsey requires $b = a$ and $Y \in [\text{depth}_X(a), X]$.

Axioms A.3 and A.4 for Ramsey Spaces

A.3 (Amalgamation)

(1) $\forall a \in \mathcal{AR}\ \forall Y \in \mathcal{S}$,

$$[d = \mathrm{depth}_Y(a) < \infty \ \rightarrow\ \forall X \in [d, Y]\ ([a, X] \neq \emptyset)],$$

(2) $\forall a \in \mathcal{AR}\ \forall X, Y \in \mathcal{S}$, letting $d = \mathrm{depth}_Y(a)$,

$$[X \leq Y \text{ and } [a, X] \neq \emptyset \ \rightarrow\ \exists Y' \in [d, Y]\ ([a, Y'] \subseteq [a, X])].$$

A.4 (Pigeonhole) Suppose $a \in \mathcal{AR}_k$ and $\mathcal{O} \subseteq \mathcal{AR}_{k+1}$. Then for every $Y \in \mathcal{S}$ such that $[a, Y] \neq \emptyset$, there exists $X \in [Y|_d, Y]$, where $d = \mathrm{depth}_Y(a)$, such that the set $\{A|_{k+1} : A \in [a, X]\}$ is either contained in $\mathcal{O}$ or is disjoint from $\mathcal{O}$.

A.3(2)-ideals

An ideal $\mathcal{I} \subseteq \mathcal{S} \times \mathcal{S}$ is a set satisfying

- $(X, Y) \in \mathcal{I} \Rightarrow X \leq Y$.
- $(X, Y) \in \mathcal{I}$ and $Z \leq X \Rightarrow (Z, Y) \in \mathcal{I}$.

$\mathcal{I}$ is an **A.3(2)-ideal** if additionally

- $\forall Y {\in} \mathcal{S}\ \forall n{<}\omega\ \exists Y' {\in} \mathcal{S}$ with $(Y', Y) \in \mathcal{I}$ and $Y'|_n = Y|_n$.
- If $(X, Y) \in \mathcal{I}$ and $a \in \mathcal{AR}^{\mathcal{X}}$, there is $Y' \in \mathcal{S}$ with $Y' \in [\mathrm{depth_Y(a), Y}]$, $(Y', Y) \in \mathcal{I}$, and $[a, Y'] \subseteq [a, X]$.

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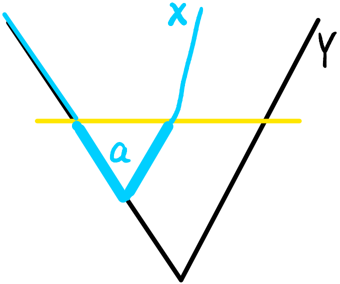

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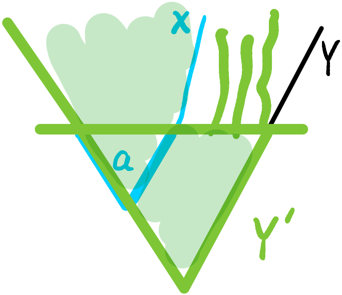
- If $(X, Y) \in \mathcal{I}$ and $a \in \mathcal{AR}^{X}$, there is $Y' \in \mathcal{S}$ with $Y' \in [\mathrm{depth_Y(a), Y}]$, $(Y', Y) \in \mathcal{I}$, and $[a, Y'] \subseteq [a, X]$.



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- If $(X, Y) \in \mathcal{I}$ and $a \in \mathcal{AR}^{x}$, there is $Y' \in \mathcal{S}$ with $Y' \in [\mathrm{depth}_{\mathrm{Y}}(\mathrm{a}), \mathrm{Y}]$, $(Y', Y) \in \mathcal{I}$, and $[a, Y'] \subseteq [a, X]$.

Abstract Ramsey Theorem from weak A.3(2)

Theorem (D., Zucker)

Suppose $(\mathcal{R}, \mathcal{S}, \leq, \leq_{\mathcal{R}})$ *satisfies axioms* **A.1**, **A.2**, **A.3(1)**, *and* **A.4**, *and suppose there is an* **A3(2)**-*ideal. Then the conclusion of the Abstract Ramsey Theorem holds.*

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Suppose $(\mathcal{R}, \mathcal{S}, \leq, \leq_{\mathcal{R}})$ satisfies axioms **A.1**, **A.2**, **A.3(1)**, *and* **A.4**, *and suppose there is an* **A3(2)***-ideal. Then the conclusion of the Abstract Ramsey Theorem holds.*

Proof follows Chapter 4 of Todorcevic, making the necessary changes.

Uses combinatorial forcing

And now, for something less forced...

Reverse Math and Big Ramsey Degrees

Anglès d'Auriac, Cholak, Dzafarov, Monin, Patey, *Milliken's tree theorem and its applications: a computability-theoretic perspective*, AMS Memoirs 2023. 136 pp.

FBRD($\mathcal{H}_3$) = "The triangle-free Henson graph has finite BRD."

Theorem (Anglès d'Auriac, Liu, Mignoty, Patey, 2022)

Carlson-Simpson's Lemma is provable in ACA_0^+. Hence, via Hubička's work, $ACA_0^+ \implies \mathrm{FBRD}(\mathcal{H}_3)$.

Theorem (Cholak, D., McCoy, 2023+)

$\mathrm{FBRD}(\mathcal{H}_3) \implies \mathrm{ACA}_0$.

References

D., *Borel sets of Rado graphs and Ramsey's Theorem*, arxXiv:1904.00266

D., *Infinite-dimensional Ramsey theory for homogeneous structures with SDAP$^+$*, arXiv:2203.00169

D.–Zucker, *Infinite-dimensional Ramsey theory for binary free amalgamation classes*, arXiv:2303.04246

An expository introduction to BRD:

D., *Ramsey theory of homogeneous structures: current trends and open problems*, Proceedings of the 2022 ICM, to appear. arXiv:2110.00655

Thank you very much!