Infinite-dimensional Ramsey theory on binary relational homogeneous structures

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Much of the work in this talk is joint with Andy Zucker.

Pigeonhole Principle

Theorem (Finite Pigeonhole Principle)

For $m < n$, if n pigeons are placed into m holes, then at least two pigeons are in the same hole.

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Figure: 10 pigeons in 9 holes, Wikimedia BenFrantzDale; McKay

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If infinitely many marbles are placed into finitely many buckets, then some bucket contains infinitely many marbles.

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Theorem (Infinite Pigeonhole Principle)

Given a coloring of the natural numbers into finitely many colors, at least one color class is infinite.

Ramsey's Theorems, 1930

Theorem (Finite Ramsey Theorem)

For $m < n$ and $2 \leq r$, there is a p large enough so that for any coloring of the m-element subsets of $\{1, \ldots, p\}$ into r colors, there is a subset of $\{1, \ldots, p\}$ of size n in which all m-element subsets have the same color.

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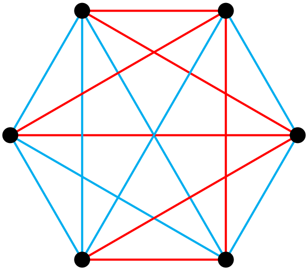
Example: $m = r = 2$, $n = 3$, $p = 6$.



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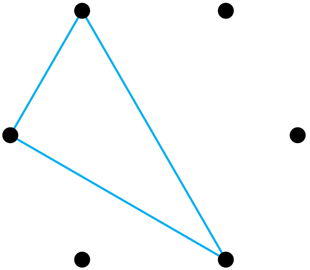

Ramsey's Theorems, 1930

Theorem (Infinite Ramsey Theorem)

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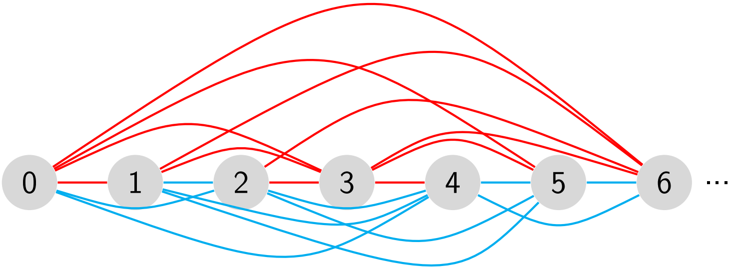

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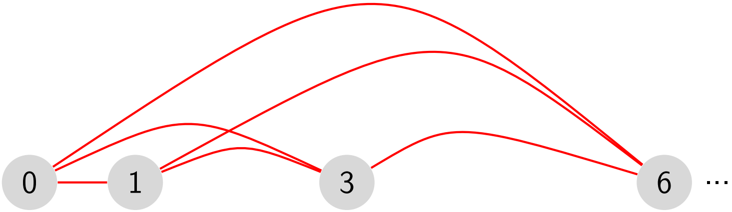

Ramsey's Theorem Re-Viewed Topologically

Def. A subset $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is **Ramsey** if each for $M \in [\mathbb{N}]^\infty$, there is an $N \in [M]^\infty$ such that $[N]^\infty \subseteq \mathcal{X}$ or $[N]^\infty \cap \mathcal{X} = \emptyset$.

Ramsey's Theorem (topological form). For any m, if $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ is a union of basic clopen sets of the form $[s, \mathbb{N}]$ where $s \in [\mathbb{N}]^m$, then $\mathcal{X}$ is Ramsey.

Coloring Infinite Sets: Topological Restrictions

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Axiom of Choice $\Longrightarrow \exists \mathcal{X} \subseteq [\omega]^\omega$ which is not Ramsey.

Solution: restrict to topologically 'definable' sets.

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Nash-Williams. Clopen sets are Ramsey.

Galvin. Open sets are Ramsey.

Galvin–Prikry. Borel sets are (completely) Ramsey.

Silver. Analytic sets are (completely) Ramsey.

Ellentuck. A set is (completely) Ramsey iff it has the property of Baire in the Vietoris (=Ellentuck) topology.

Ellentuck Theorem: The Best Infinite-Dimensional Thm.

Ellentuck topology: refines the metric topology with basic open sets

$$[s, A] = \{B \in [\omega]^\omega : s \sqsubset B \subseteq A\}.$$

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A set $\mathcal{X} \subseteq [\mathbb{N}]^\infty$ satisfies

$(*)$ $\quad \forall [s, A] \;\; \exists B \in [s, A]$ *such that* $[s, B] \subseteq \mathcal{X}$ *or* $[s, B] \cap \mathcal{X} = \emptyset$

iff $\mathcal{X}$ has the property of Baire with respect to the Ellentuck topology.

$(*)$ is called completely Ramsey by Galvin–Prikry and Ramsey by Todorcevic.

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Topological Ramsey spaces: Points are infinite sequences, topology is induced by finite heads and infinite tails, and every subset with the property of Baire satisfies $(*)$.
(Carlson–Simpson 1990; Todorcevic 2010.)

Part of Question 11.2 of Kechris–Pestov–Todorcevic

Develop infinite-dimensional Ramsey theory* for the

(i) Rationals;

(ii) Ordered Rado graph;

(iii) *k*-clique-free ordered Henson graphs;

(iv) Random $\mathcal{A}$-free ordered hypergraph, where $\mathcal{A}$ is a set of finite irreducible ordered structures;

(v) Ordered rational Urysohn space;

(vi) $\aleph_0$-dimensional vector space over a finite field with the canonical ordering;

(vii) The countable atomless Boolean algebra with the canoncial ordering.

$*$ A successful topological characterization should recover big Ramsey degrees exactly.

Structural Ramsey Theory

Structural Ramsey Theory

- (finite) Colorings of finite structures within finite structures.
- (finite-dimensional) Colorings of finite structures within infinite structures.
- (infinite-dimensional) Colorings of infinite structures within infinite structures.

Finite Structural Ramsey Theory

For structures $\mathbf{A}, \mathbf{B}$, write $\mathbf{A} \le \mathbf{B}$ iff $\mathbf{A}$ embeds into $\mathbf{B}$.

$\binom{\mathbf{B}}{\mathbf{A}}$ denotes the set of all copies of $\mathbf{A}$ in $\mathbf{B}$.

A class $\mathcal{K}$ of finite structures has the **Ramsey Property** if given $\mathbf{A} \le \mathbf{B}$ in $\mathcal{K}$ and r, there is $\mathbf{C} \in \mathcal{K}$ so that

$$\forall \chi : \binom{\mathbf{C}}{\mathbf{A}} \to r \quad \exists \mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}, \ \chi \restriction \binom{\mathbf{B}'}{\mathbf{A}} \text{ is constant.}$$

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Lots of work done! (e.g., Nešetřil–Rödl, Hubička–Nešetřil)

Examples and correspondence with dynamics

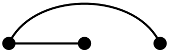
Classes of finite structures with the Ramsey Property:

- linear orders (Ramsey)
- Boolean algebras (Graham-Rothschild)
- ordered graphs, k-clique-free graphs, hypergraphs,
- ordered free amalgamation classes (Nešetřil-Rödl).

Theorem (Kechris, Pestov, Todorcevic, 2005)

A Fraïssé class $\mathcal{K}$ has the Ramsey property iff the automorphism group of its Fraïssé limit is extremely amenable.

Example: Colorings of copies of a finite graph



A

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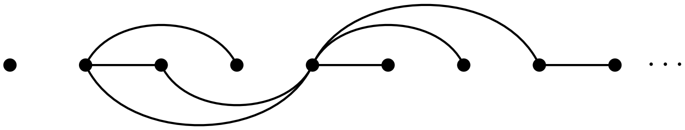

B

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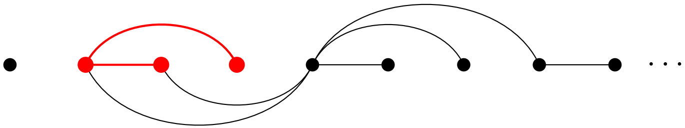
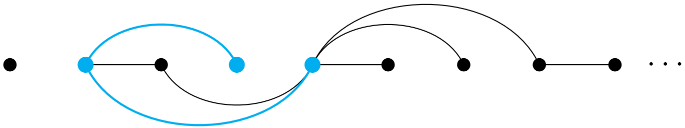


A red copy of **A** in **B**

Example: Colorings of copies of a finite graph



A blue copy of **A** in **B**

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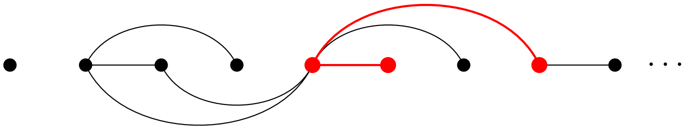

Another red copy of **A** in **B**

What if we want $\mathbf{B} = \mathbf{C}$ infinite?

Ramsey theory on infinite structures

(Infinite) Homogeneous Structures

A structure **K** is **homogeneous** if every isomorphism between two finite induced substructures of **K** extends to an automorphism of **K**.

Homogeneous structures are Fraïssé limits of Fraïssé classes. Examples include the

- $(\mathcal{R}, E)$ Rado graph
- $(\mathcal{H}_k, E)$ k-clique-free Henson graphs, $k \geq 3$
- generic k-partite graph
- generic digraph
- random graph with Red and Blue edges omitting RRB and RBB triangles and Red 4-cliques
- generic partial order
- rationally ordered versions: $(\mathcal{R}, E, <)$, $(\mathcal{H}_k, E, <)$, ...
- Free superpositions of the above

Big Ramsey Degrees

Let $\mathcal{K}$ be a Fraïssé class of finite structures with limit $\mathbf{K}$.

$\mathbf{K}$ has **finite big Ramsey degrees** if for each finite $\mathbf{A} \in \mathcal{K}$, $\exists t$ such that $\forall r$, $\forall\ \chi : \binom{\mathbf{K}}{\mathbf{A}} \to r$, $\exists \mathbf{K}' \in \binom{\mathbf{K}}{\mathbf{K}}$ such that $|\chi \restriction \binom{\mathbf{K}'}{\mathbf{A}}| \leq t$.

$$\mathbf{K} \to (\mathbf{K})^{\mathbf{A}}_{r,t}$$

The **big Ramsey degree** of $\mathbf{A}$ in $\mathbf{K}$ $= \mathrm{BRD}(\mathbf{A}, \mathbf{K}) = \mathrm{BRD}(\mathbf{A})$ is the least such t.

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- (Hjorth 2008) If $|\mathrm{Aut}(\mathbf{K})| > 1$, then $\mathcal{K}$ has some BRD > 1.

BRD's are really about the optimal structural expansions for which Ramsey's Theorem holds. (canonical partitions)

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Theorem (Zucker, 2019)

It $\mathbf{K}$ has a big Ramsey structure, then $\mathrm{Aut}(\mathbf{K})$ admits a universal completion flow.

Big Ramsey Degree results, a sampling

- 1933. BRD(Pairs, $\mathbb{Q}$) ≥ 2. (Sierpiński)
- 1975. BRD(Edge, $\mathcal{R}$) ≥ 2. (Erdős, Hajnal, Pósa)
- 1979. $(\mathbb{Q}, <)$: All BRD computed. (D. Devlin)
- 1986. BRD(Vertex, $\mathcal{H}_3$) $= 1$. (Komjáth, Rödl)
- 1989. BRD(Vertex, $\mathcal{H}_n$) $= 1$. (El-Zahar, Sauer)
- 1996. BRD(Edge, $\mathcal{R}$) $= 2$. (Pouzet, Sauer)
- 1998. BRD(Edge, $\mathcal{H}_3$) $= 2$. (Sauer)
- 2006, 2008. The Rado graph: All BRD characterized; computed. (Laflamme, Sauer, Vuksanović); (J. Larson)
- 2008. Ultrametric spaces with finite distance set: All BRD characterized. (Nguyen Van Thé)
- 2010. Dense Local Order $\mathbf{S}(2)$: All BRD computed. Also $\mathbb{Q}_n$. (Laflamme, Nguyen Van Thé, Sauer)

∞ Structural RT via coding trees and forcing (arxiv dates)

- 2017. Triangle-free Henson graphs: FBRD foreshadowing ∞-diml Exact bounds via small tweak in 2020. (D.) and independently (BDHKVZ)
- 2019. k-clique-free Henson graphs: Upper Bounds. (D.)
- 2019. ∞-dimensional RT for Borel sets of Rado graphs. (D.)
- 2020. Binary rel. Forb($\mathcal{F}$): Upper Bounds. (Zucker)
- 2020. Exact BRD for binary SDAP$^+$ structures. (Coulson, D., Patel)
- 2021. Binary rel. Forb($\mathcal{F}$): Exact BRD. (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker)
- 2022. ∞-dimensional RT structures with SDAP$^+$. recovers Exact BRD. (D.)
- 2023+. ∞-dimensional RT for finitely constrained binary FAP. recovers Exact BRD. (D., Zucker)

Developments not using forcing (arxiv dates)

- 2018. Certain homogeneous metric spaces: FBRD. (Mašulović) category th.
- 2019. 3-uniform hypergraphs: FBRD. (Balko, Chodounský, Hubička, Konečný, Vena) Milliken Theorem.
- 2020. Circular directed graphs: Exact BRD Computed. (Dasilva Barbosa) category theory.
- 2020. Homogeneous partial order: FBRD. (Hubička) Ramsey space of parameter words. First non-forcing proof for $\mathcal{H}_3$.
- 2021. Homogenous graphs with forbidden cycles (metric spaces): FBRD. (Balko, Chodounský, Hubička, Konečný, Nešetřil, Vena) param. words.
- 2023. Homogeneous partial order: Exact BRD. (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker) parameter words.
- 2023. Infinite languages, unrestricted structures: FBRD. (Braunfeld, Chodounský, de Rancourt, Hubička, Kawach, Konečný. Laver Theorem.
- 2023+. Many Forb($\mathcal{F}$), all arities, and more: FBRD. (BCDHKNVZ) New methods.
- 2023+. Pseudotrees. (Chodounský, D., Eskew, Weinert)

What comprises a big Ramsey degree?

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It begins with enumerating the vertices of the structure.

Coding Tree of 1-types for $(\mathbb{Q}, <)$

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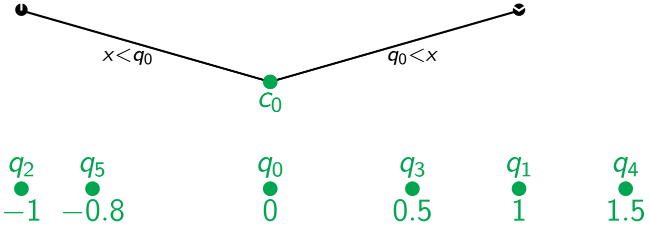

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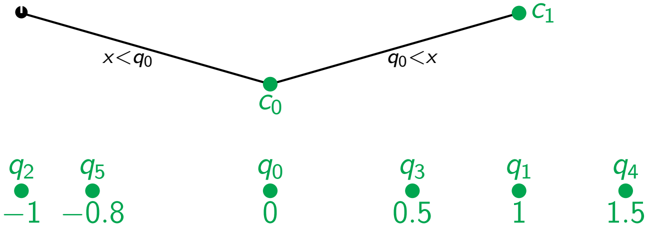

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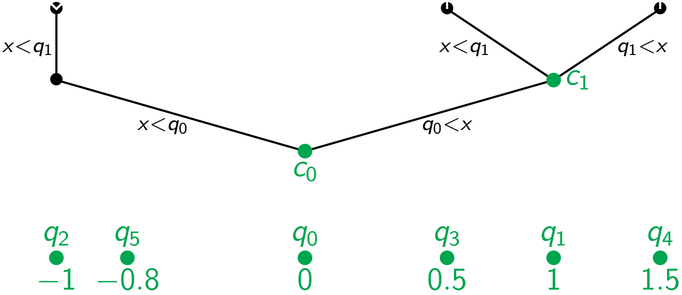

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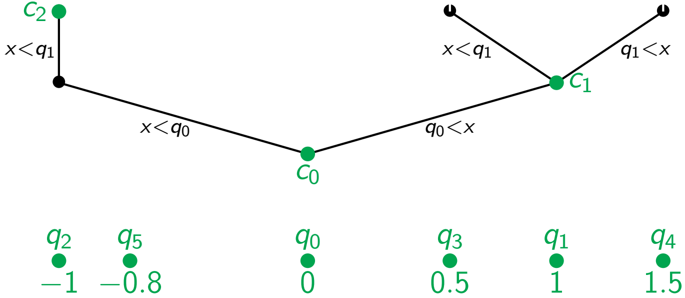

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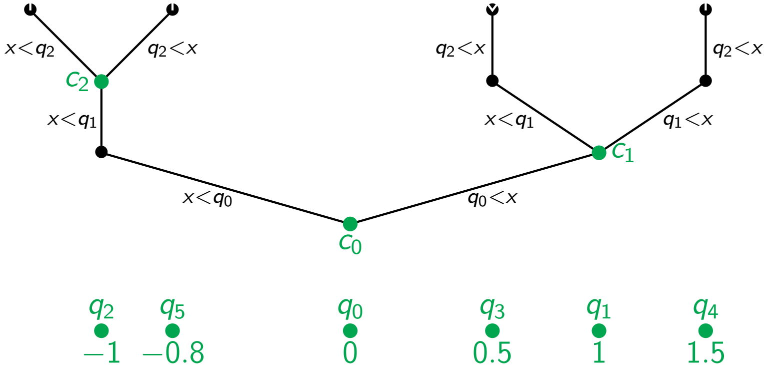

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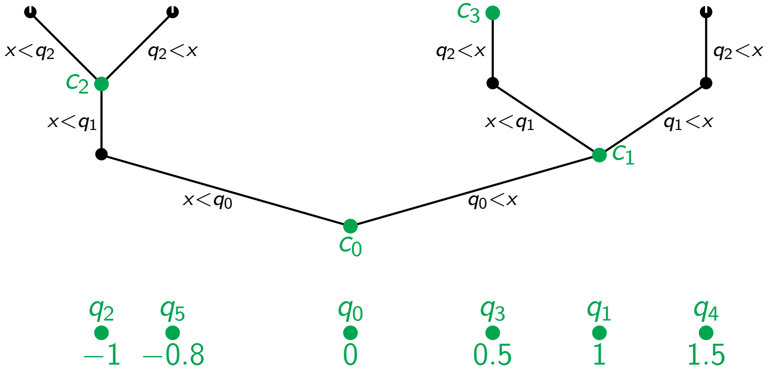

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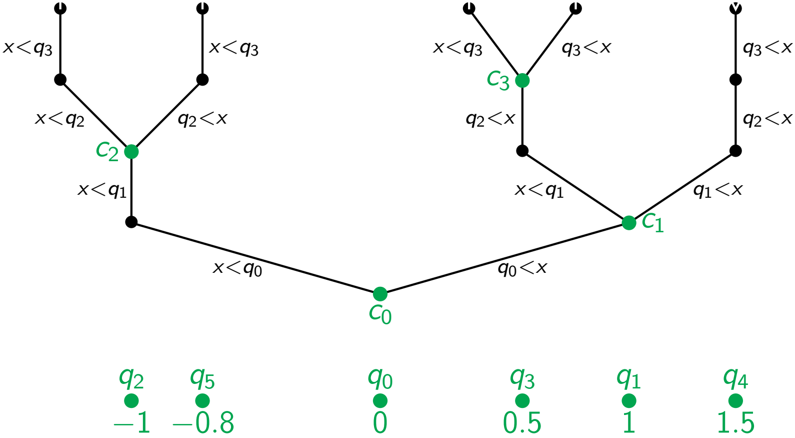

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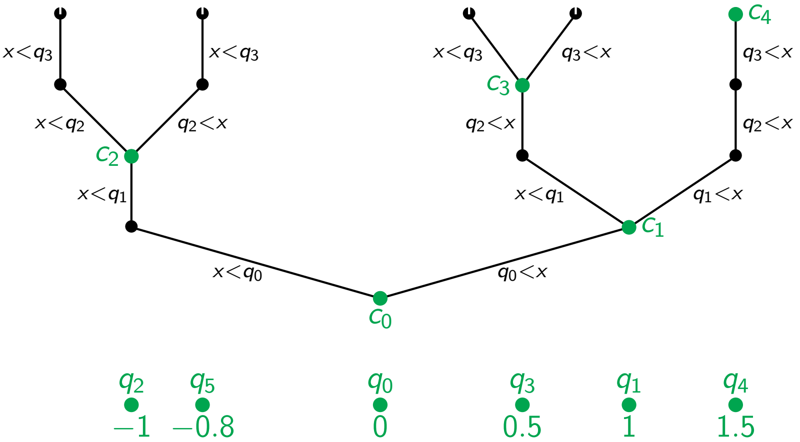

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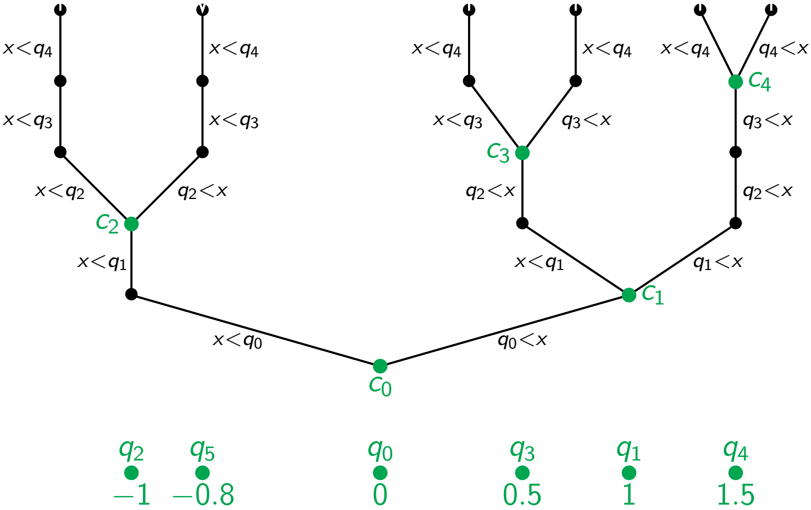

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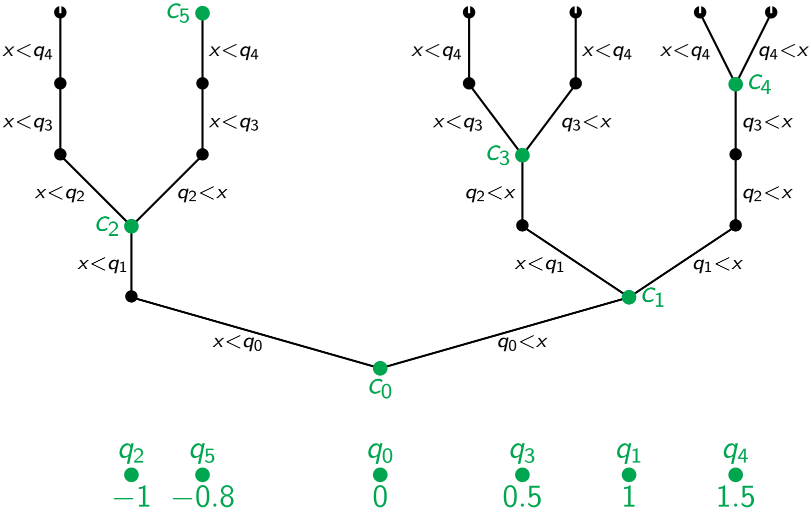

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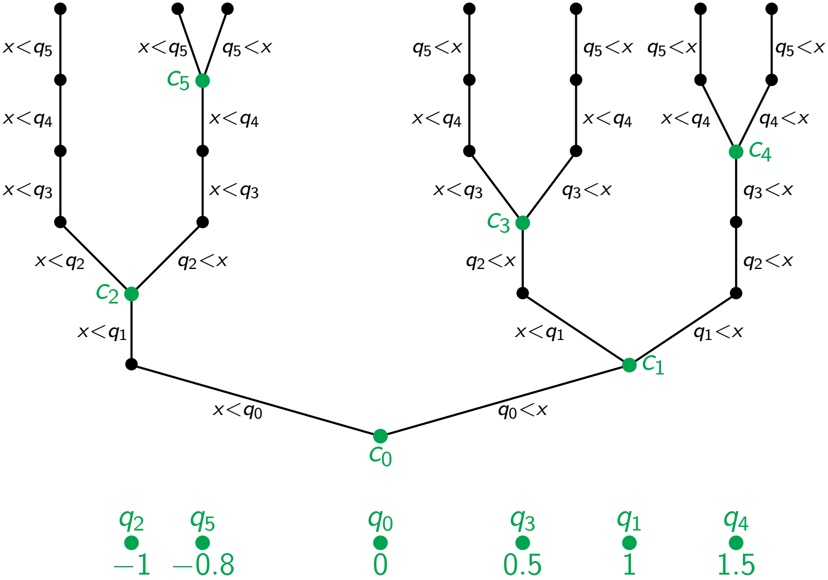
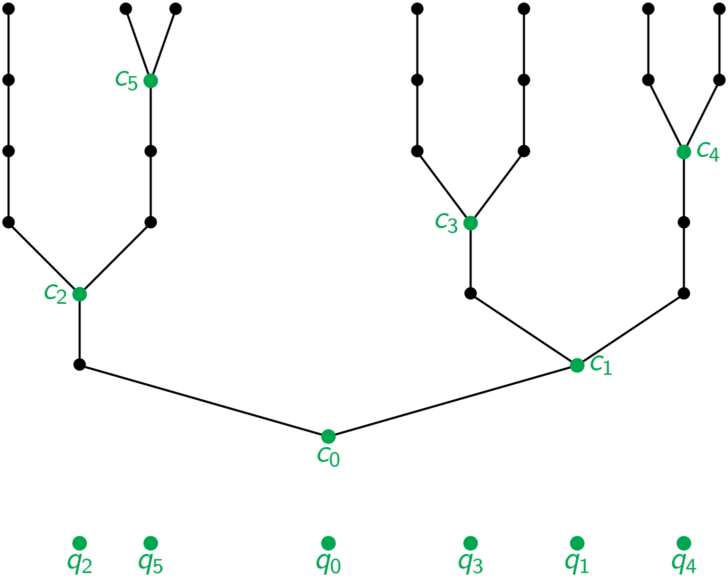


Sierpiński's Coloring



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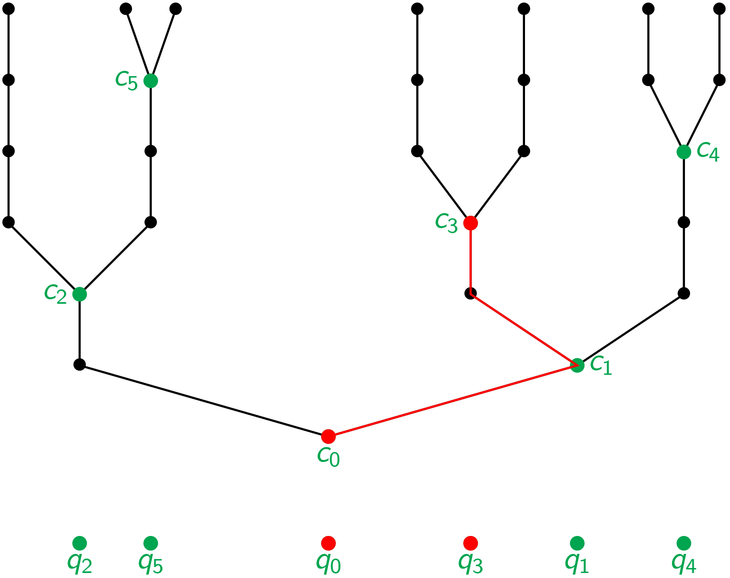

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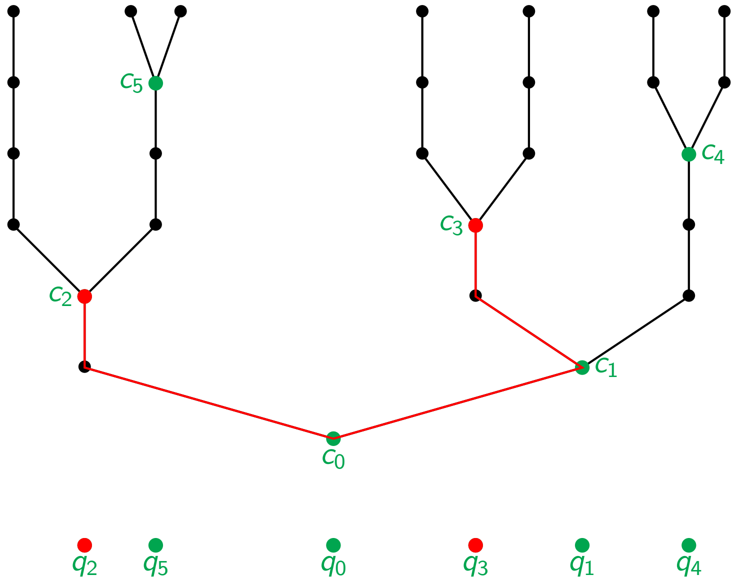

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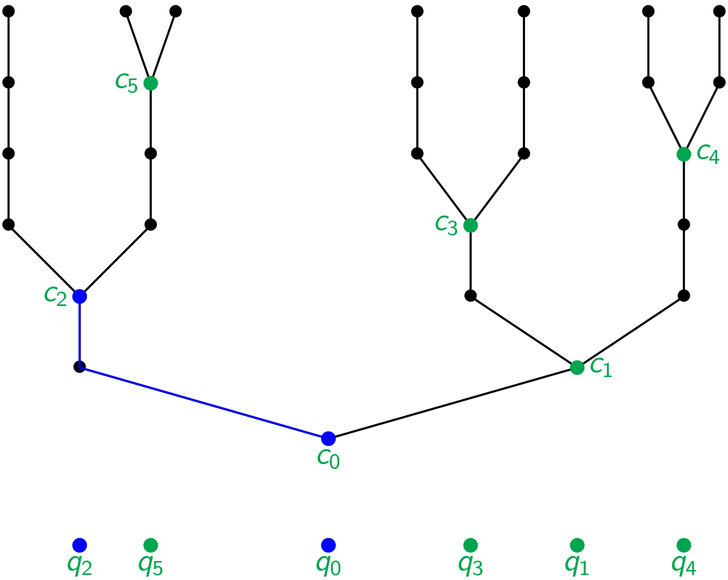

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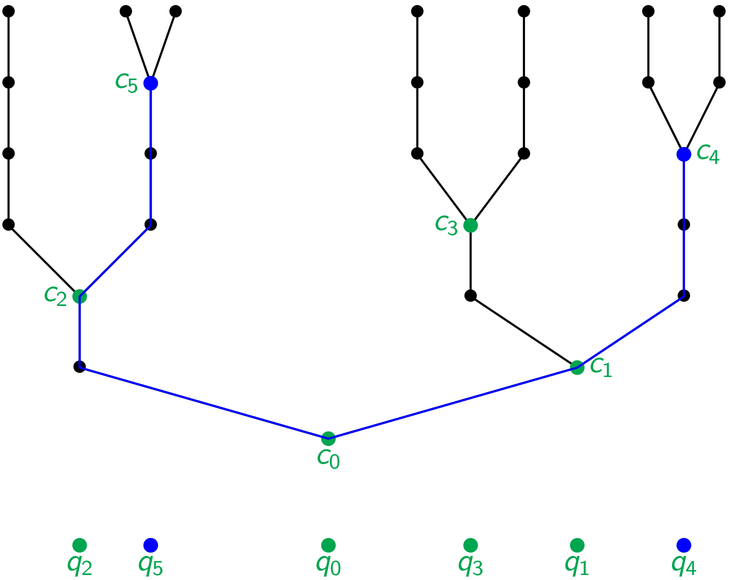
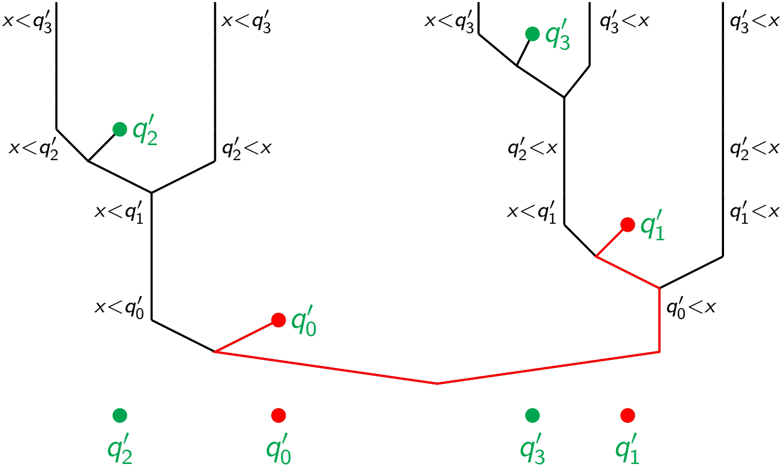


Devlin's Diagonal Antichains and Exact Degrees



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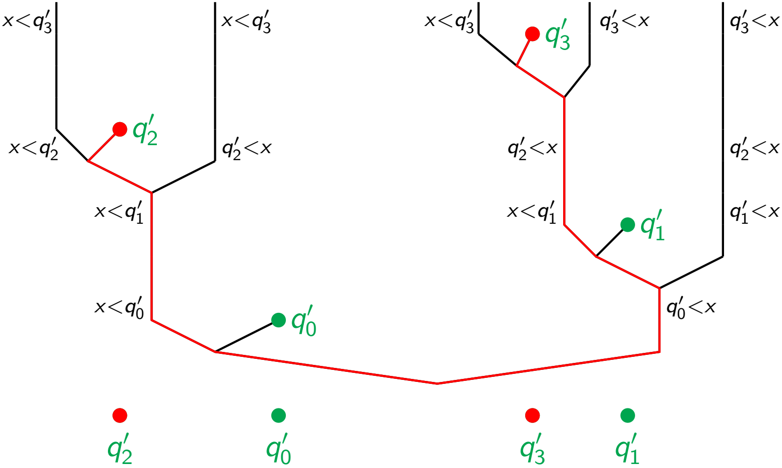

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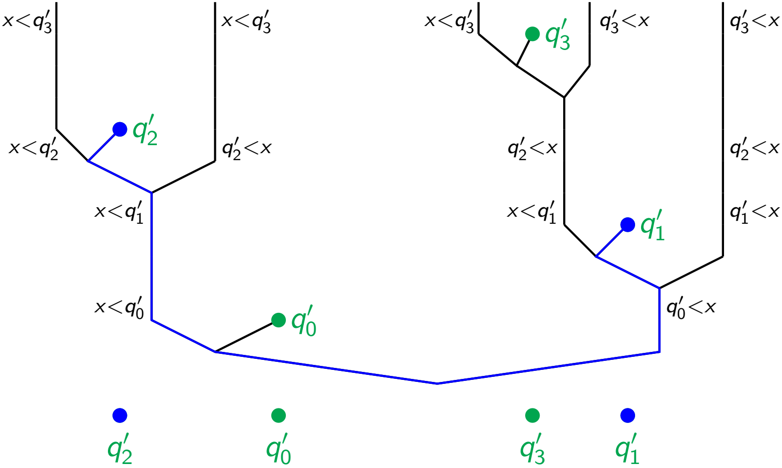
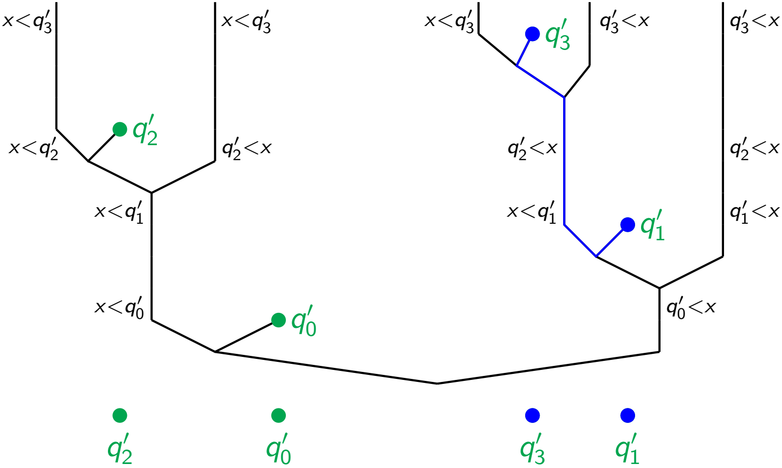
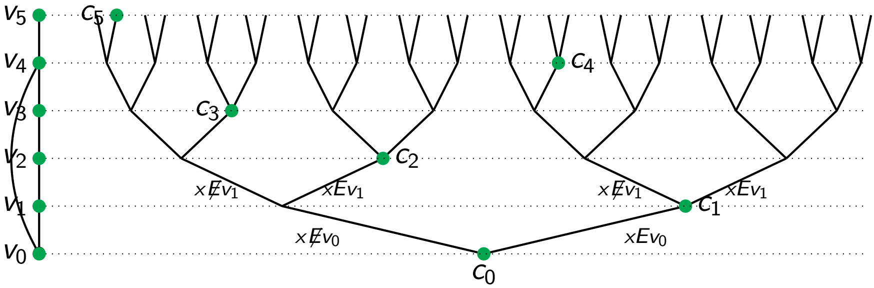


Devlin's Diagonal Antichains and Exact Degrees



Coding Tree of 1-types for the Rado Graph, $\mathcal{R}$



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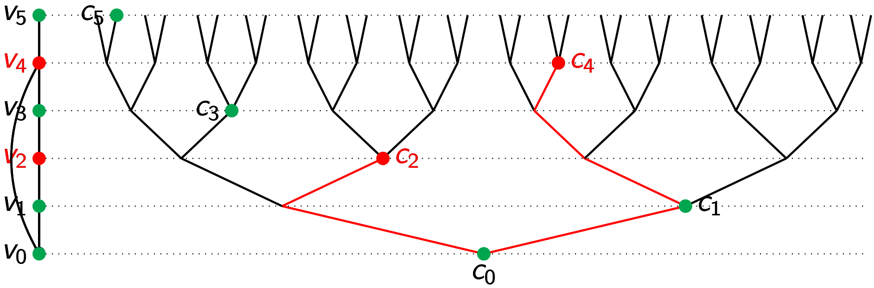
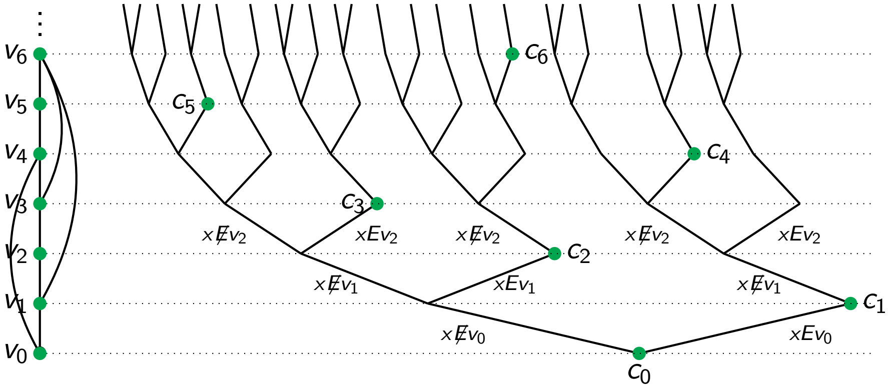


Coding Tree of 1-types for $\mathcal{H}_3$

Enumerating the vertices of $\mathcal{H}_3$ induces the tree possibilities.



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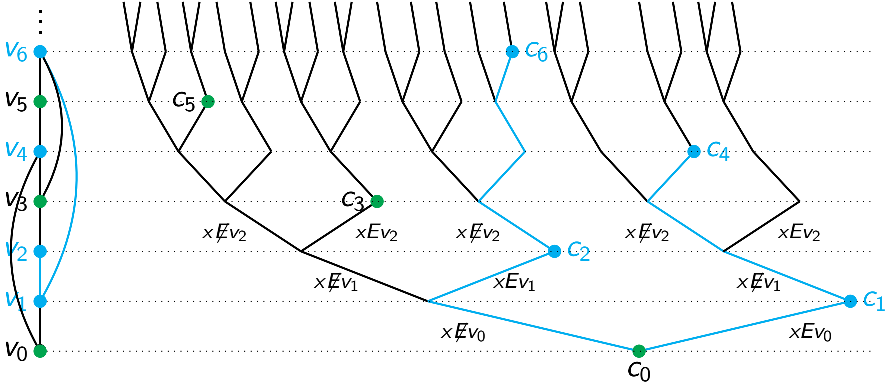

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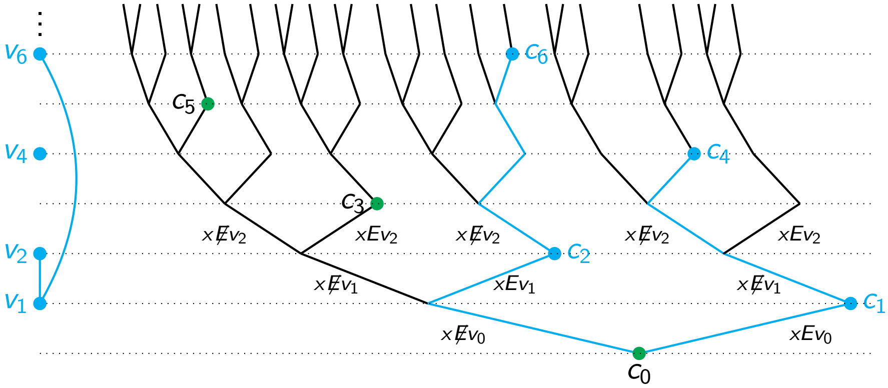

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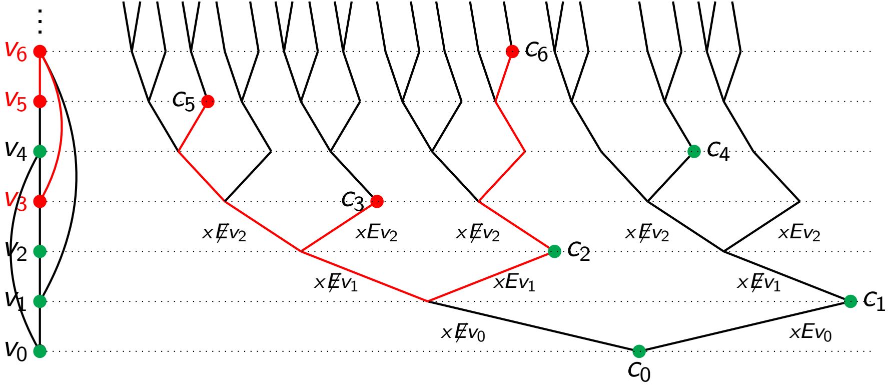

This is an antichain, even diagonal.

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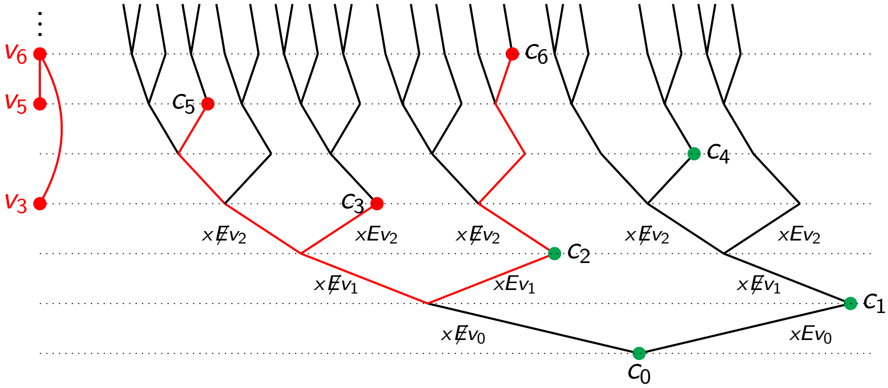

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Finitely constrained binary relational FAP classes

A structure is **irreducible** if any two vertices are in some relation: e.g., finite clique, finite tournament, triangle with 2 red edges and one green edge.

Free amalgamation classes are exactly of the form Forb($\mathcal{F}$), where $\mathcal{F}$ is a set of finite irreducible structures.

It is **finitely constrained, binary FAP** if the language consists of finitely many relations of arity at most two and $\mathcal{F}$ is finite.

Finitely constrained binary FAP classes

Theorem (Dobrinen, 2017, 2019)

The k-clique-free Henson graphs have finite big Ramsey degrees.

Theorem (Zucker, 2020)

All finitely constrained binary FAP classes have finite big Ramsey degrees.

Theorem (Balko, Chodounský, Dobrinen, Hubička, Konečný, Vena, Zucker, 2021)

We characterize the exact big Ramsey degrees of all finitely constrained binary FAP classes.

Big Ramsey Degree Characterizations: Diaries

Big Ramsey degrees of a binary relational homogeneous structure **K** are characterized via enumerating the universe of **K** and forming the coding tree of 1-types and

I. Diagonal antichains (in the coding tree of 1-types);

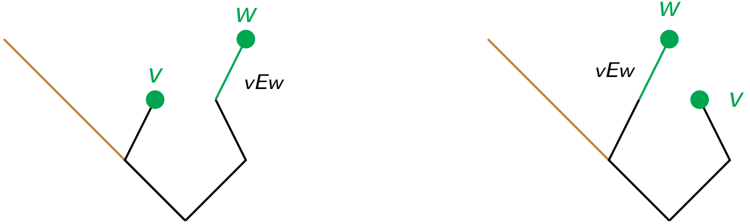
II. Passing types;

III. Forbidden substructures also include

- I(a). Controlled splitting levels;
- II(a). Controlled coding triples;
- III(a). Maximal paths;
- III(b). Essential age-change levels (incremental changes in how much of a forbidden substructure is coded).

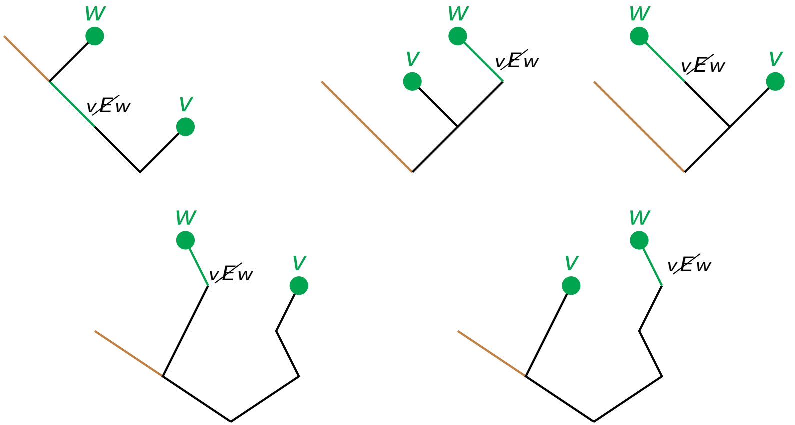
$T(\text{Edge}, \mathcal{H}_3) = 2$

These are the unavoidable patterns representing edges.



$T(\text{Non-Edge}, \mathcal{H}_3) = 5$

These are the unavoidable patterns representing non-edges.



Infinite-Dimensional Structural Ramsey Theory

$$\mathbf{K} \rightarrow^* (\mathbf{K})^{\mathbf{K}}$$

- Well-ordering **K** induces a metric topology, like Baire space.

- Any infinite-dimensional structural Ramsey theory must start by fixing a diary and then working with the space of all subcopies of that diary.

Abstract Ramsey Theorem (∞-diml Ramsey Theory)

Theorem (Todorcevic)

Suppose we are given a structure $(\mathcal{R}, \mathcal{S}, \leq, \leq_{\mathcal{R}})$ with finite restrictions maps satisfying Axioms A.1 to A.4, and that $\mathcal{S}$ is closed. Then the field of $\mathcal{S}$-Ramsey subsets of $\mathcal{R}$ is closed under the Souslin operation and it coincides with the field of $\mathcal{S}$-Baire subsets of $\mathcal{R}$.

$$\mathcal{R} = \mathcal{S} \implies \text{Abstract Ellentuck Theorem}$$

So if we could just show that our spaces of subcopies of **K** satisfy these four axioms, we'd be done.

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$\mathcal{R} = \mathcal{S} \implies$ Abstract Ellentuck Theorem

So if we could just show that our spaces of subcopies of **K** satisfy these four axioms, we'd be done. BUT

- BRDs preclude working with spaces of ALL subcopies of **K**.
- A.3(2) generally usually fails for Fraïssé structures.

Part of Question 11.2 of Kechris–Pestov–Todorcevic

Develop infinite-dimensional Ramsey theory* for the

(i) Rationals;

(ii) Ordered Rado graph;

D. 2022 SDAP$^+$ structures

(iii) k-clique-free ordered Henson graphs; D.-Zucker 2023 all bin. FAP

(iv) Random $\mathcal{A}$-free ordered hypergraph, where $\mathcal{A}$ is a set of finite irreducible ordered structures;

(v) Ordered rational Urysohn space;

(vi) $\aleph_0$-dimensional vector space over a finite field with the canonical ordering; Impossible for $\mathbb{F}_p$, $p \geq 3$, L, NVT, P, S 2011

(vii) The countable atomless Boolean algebra with the canoncial ordering.

* A successful topological characterization should recover big Ramsey degrees exactly.

∞-diml RT for binary finitely constrained FAP classes

Theorem (D., Zucker)

Let **K** *be a finitely constrained homogeneous structure with free amalgamation and finitely many relations of arity* ≤ 2. *Then* **K** *has an infinite-dimensional Ramsey theory which directly recovers the exact big Ramsey degrees in (BCDHKVZ 2021).*

Proof Outline:

(1) Prove that a weaker version of A.3 suffices to guarantee the Abstract Ramsey Theorem.

(2) Show that certain two-sorted spaces of diaries satisfy weakened A.3(2).

(3) "Force" a Pigeonhole Lemma for colorings of copies of a given level set.

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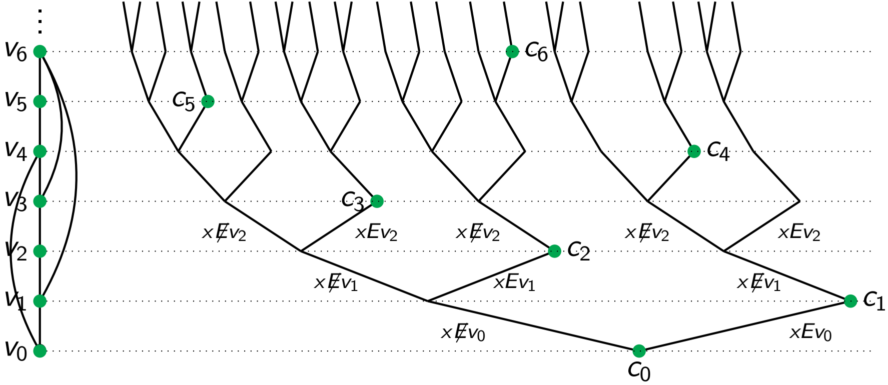
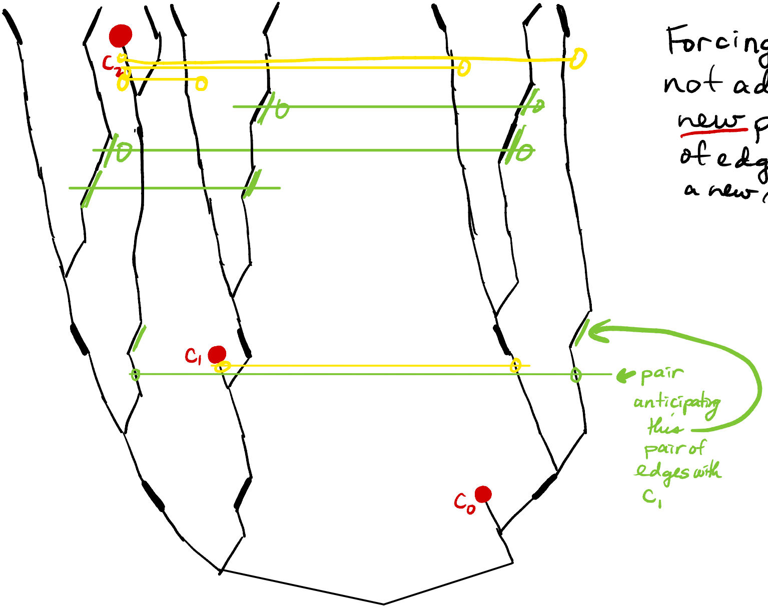


A Strong Diary Δ for $\mathcal{H}_3$



Forcing must not add new pairs of edges with a new vertex.

$\mathcal{S}$-Baire and $\mathcal{S}$-Ramsey sets

For $X \in \mathcal{S}$ and a finite approximation a to some member of $\mathcal{R}$,

$$[a, X] = \{A \in \mathcal{R} : A \leq_{\mathcal{R}} X \text{ and } a \sqsubset A\}$$

A set $\mathcal{X} \subseteq \mathcal{R}$ is **$\mathcal{S}$-Baire** if for every non-empty basic open set $[a, X]$ there is an $a \sqsubseteq b \in \mathcal{AR}$ and $Y \leq X$ in $\mathcal{S}$ such that $[b, Y] \neq \emptyset$ and $[b, Y] \subseteq \mathcal{X}$ or $[b, Y] \subseteq \mathcal{X}^c$.

$\mathcal{S}$-Ramsey requires $b = a$ and $Y \in [\text{depth}_X(a), X]$.

Axioms A.3 and A.4 for Ramsey Spaces

A.3 (Amalgamation)

(1) $\forall a \in \mathcal{AR}\ \forall Y \in \mathcal{S}$,

$$[d = \mathrm{depth}_Y(a) < \infty \ \rightarrow\ \forall X \in [d, Y]\ ([a, X] \neq \emptyset)],$$

(2) $\forall a \in \mathcal{AR}\ \forall X, Y \in \mathcal{S}$, letting $d = \mathrm{depth}_Y(a)$,

$$[X \leq Y \text{ and } [a, X] \neq \emptyset \ \rightarrow\ \exists Y' \in [d, Y]\ ([a, Y'] \subseteq [a, X])].$$

A.4 (Pigeonhole) Suppose $a \in \mathcal{AR}_k$ and $\mathcal{O} \subseteq \mathcal{AR}_{k+1}$. Then for every $Y \in \mathcal{S}$ such that $[a, Y] \neq \emptyset$, there exists $X \in [Y|_d, Y]$, where $d = \mathrm{depth}_Y(a)$, such that the set $\{A|_{k+1} : A \in [a, X]\}$ is either contained in $\mathcal{O}$ or is disjoint from $\mathcal{O}$.

A.3(2)-ideals

An ideal $\mathcal{I} \subseteq \mathcal{S} \times \mathcal{S}$ is a set satisfying

- $(X, Y) \in \mathcal{I} \Rightarrow X \leq Y$.
- $(X, Y) \in \mathcal{I}$ and $Z \leq X \Rightarrow (Z, Y) \in \mathcal{I}$.

$\mathcal{I}$ is an **A.3(2)-ideal** if additionally

- $\forall Y \in \mathcal{S}\ \forall n < \omega\ \exists Y' \in \mathcal{S}$ with $(Y', Y) \in \mathcal{I}$ and $Y'|_n = Y|_n$.
- If $(X, Y) \in \mathcal{I}$ and $a \in \mathcal{AR}^{\mathcal{X}}$, there is $Y' \in \mathcal{S}$ with $Y' \in [\mathrm{depth_Y(a), Y}]$, $(Y', Y) \in \mathcal{I}$, and $[a, Y'] \subseteq [a, X]$.

Abstract Ramsey Theorem from weak A.3(2)

Theorem (D., Zucker)

Suppose $(\mathcal{R}, \mathcal{S}, \leq, \leq_{\mathcal{R}})$ *satisfies axioms* **A.1**, **A.2**, **A.3(1)**, *and* **A.4**, *and suppose there is an* **A3(2)***-ideal. Then the conclusion of the Abstract Ramsey Theorem holds.*

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Along with our other work, we obtain an analogue of Ellentuck's Thm. for natural spaces of subcopies of $\mathbf{K}$, for all fin. constr. binary FAP $\mathbf{K}$.

References

D.–Zucker, *Infinite-dimensional Ramsey theory for binary free amalgamation classes*, arXiv:2303.04246

D., *Ramsey theory of homogeneous structures: current trends and open problems*. Proceedings of the 2022 International Congress of Mathematicians (to appear). arXiv:2110.00655

Thank you very much!