Cofinal types of ultrafilters on measurable cardinals

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Tukey reduction on partial orders

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A function $g : P \to Q$ is unbounded if for every unbounded subset $X \subseteq P$, $g''X$ is unbounded in Q.

P is Tukey reducible to Q, $P \leq_T Q$, if there is a cofinal map $f : Q \to P$ or, equivalently, an unbounded map $g : P \to Q$.

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$P \equiv_T Q$ iff $P \leq_T Q$ and $Q \leq_T P$.

When P and Q are directed posets, then $P \equiv_T Q$ iff there is a third directed poset into which they both embed cofinally.

$[P]_T = \{Q : Q \equiv_T P\}$ is the cofinal type of P.

Tukey reduction on ultrafilters

Let κ be an infinite regular cardinal.

Given an ultrafilter U on κ, $(U, \supseteq)$ is a directed partial order.

$X \subseteq U$ is a **filter base** for U if for each $A \in U$ there is a $B \in X$ such that $A \supseteq B$.

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A map $f : U \to V$ is **cofinal** if for each filter base $X \subseteq U$, $f''X$ is a filter base for V.

V is **Tukey reducible** to U, $U \geq_T V$, iff there is a cofinal map from U to V.

Cofinal Types

$U \geq_T V$ iff there is a map $f : U \to V$ taking each filter base of U to a filter base of V.

$$U \equiv_T V \iff U \leq_T V \text{ and } V \leq_T U$$

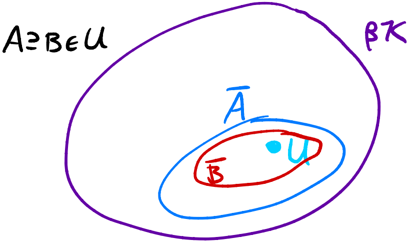
$\equiv_T$ is an equivalence relation on the set of ultrafilters on κ.

$[U]_T =$ the set of ultrafilters V on κ such that $V \equiv_T U$.
$=$ the **cofinal type** or **Tukey type** of U.

Tukey reducibility has its roots in the development of the notion of convergence in general topology.

Nets = directed partial orders.

Neighborhood bases in $\beta\kappa$.



$\mathcal{U} \in \beta\kappa$

$\forall A \in \mathcal{U}$, $\bar{A} := \{\mathcal{V} \in \beta\kappa : A \in \mathcal{V}\}$ is an open nbhd of $\mathcal{U}$.

Tukey coarsens Rudin-Keisler

$U \geq_{RK} V$ iff there is an $h : \kappa \to \kappa$ such that

$$h^*(U) := \{A \subseteq \kappa : h^{-1}(A) \in U\} = V;$$

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- $U \geq_{RK} V \implies U \geq_T V$
- Tukey equivalence coursens RK equivalence.

Maximum Cofinal Type for Ultrafilters on ω

$([\mathfrak{c}]^{<\omega}, \subseteq)$ is Tukey-top among directed partial orders of size $\mathfrak{c}$.

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$([\mathfrak{c}]^{<\omega}, \subseteq)$ is Tukey-top among directed partial orders of size $\mathfrak{c}$.

Theorem. (Isbell 1965, Juhasz 1967) There is an ultrafilter on ω which is Tukey top.

Construction uses an independent family of size $\mathfrak{c}$.

Isbell's Problem

Fact. An ultrafilter U on ω is not Tukey top iff

$$\forall \langle A_i : i < \mathfrak{c} \rangle \in [U]^{\mathfrak{c}} \ \ \exists I \in [\mathfrak{c}]^{\omega} \ \bigcap_{i \in I} A_i \in U$$

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Consistently NO. • (Milovich 08) using $\Diamond$.

- (D.-Todorcevic 11) p-points (and hence Ramsey ultrafilters), stable ordered union ultrafilters, Fubini iterates, basically generated ultrafilters are all not Tukey top.
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Isbell's Problem Today: Is there a model of ZFC in which all ultrafilters on ω are Tukey maximum?

Cofinal types of ultrafilters on ω

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(3) Other conditions guaranteeing an ultrafilter is not of the maximum cofinal type (Milovich 08), (D.-Todorcevic 11), (Raghavan-Todorcevic 12), (Blass-D.-Raghavan 15), (D. 16);

Cofinal types of ultrafilters on ω

(4) Embeddings of various partial orders: $2^{\mathfrak{c}}$ incomparable selective ultrafilters, and p-points (D.-Todorcevic 11), (Raghavan-Todorcevic 12), $\mathcal{P}(\omega)/\mathrm{fin}$ (Raghavan-Shelah 17), long lines (Kuzeljevic-Raghavan 18), (Raghavan-Verner 19);

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(6) Finding the exact structure of the cofinal types, including the precise structure of the Rudin-Keisler classes inside them, below Ramsey ultrafilters (Raghavan-Todorcevic 12), below certain p-points satisfying weak partition relations (D.-Todorcevic 14), (D.-Todorcevic 15), (D.-Mijares-Trujillo 17), and below non-p-points forced by $\mathcal{P}(\omega^k)/\text{fin}^{\otimes k}$ (D. 16).

Ultrafilters on κ: Galvin and Tukey

Special Ultrafilters on κ

Let U be an ultrafilter over a regular cardinal κ. U is

1. uniform if for every $X \in U$, $|X| = \kappa$.
2. λ-complete if U is closed under $< \lambda$ intersections.
3. normal if U is closed under diagonal intersection: if $\langle A_i \mid i < \kappa \rangle \subseteq U$, then $\Delta_{i<\kappa} A_i \in U$.

$$\Delta_{i<\kappa} A_i := \{\nu < \kappa \mid \forall i < \nu \, (\nu \in A_i)\}$$

4. Ramsey if for any function $f : [\kappa]^2 \to 2$ there is an $X \in U$ such that $f \restriction [X]^2$ is constant.
5. selective if for every function $f : \kappa \to \kappa$, there is an $X \in U$ such that $f \restriction X$ is either constant or one-to-one.

$$\text{normal} \implies \text{Ramsey} = \text{selective}$$

Special Ultrafilters on κ

A function $f : \kappa \to \kappa$ is almost 1-1 (mod U) if there is an $A \in U$ such that for every $\gamma < \kappa$, $|f^{-1}[\gamma] \cap A| < \kappa$.

U is a

1. *p*-point if whenever $f : \kappa \to \kappa$ is not constant (mod U) then f is almost 1-1 (mod U).
2. *q*-point if every function $f : \kappa \to \kappa$ which is almost 1-1 (mod U) is 1-1 (mod U).

$$\text{normal} \implies \text{selective} = \text{p-point} + \text{q-point}$$

• selective $\iff$ RK-minimal among uniform ultrafilters
$\iff$ RK equivalent to a normal ultrafilter

Measurable Cardinals and Ultrafilters

The following are equivalent:

1. κ is measurable;
2. There is a κ-complete ultrafilter over κ;
3. There is a Ramsey ultrafilter on κ;
4. There is a normal ultrafilter on κ;
5. κ is the critical point of some nontrivial elementary embedding $j : V \to M$.

U on κ is normal if U is closed under diagonal intersection:
if $\langle A_i \mid i < \kappa \rangle \subseteq U$, then $\Delta_{i<\kappa} A_i := \{\nu < \kappa \mid \forall i < \nu \, (\nu \in A_i)\} \in U$.

The Galvin Property and Tukey Non-Top

For U an ultrafilter on κ and $\lambda \leq \nu$, $\mathrm{Gal}(U, \lambda, \nu)$ holds iff

$$\forall \langle A_i : i < \nu \rangle \in [U]^\nu \;\; \exists I \in [\nu]^\lambda \; \bigcap_{i \in I} A_i \in U$$

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Theorem. (Galvin, 1973) Suppose $\kappa^{<\kappa} = \kappa$. Then for every normal filter U on κ, $\mathrm{Gal}(U, \kappa, \kappa^+)$ holds.

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Theorem. (Benhamou-D.) $\lambda \leq \kappa$, λ inaccessible, U a λ-complete ultrafilter over κ. Then $(U, \supseteq) \equiv_T ([2^\kappa]^{<\lambda}, \subseteq)$ iff $\neg\,\mathrm{Gal}(U, \lambda, 2^\kappa)$.

- If κ is measurable, then the κ-complete ultrafilters over κ which are Tukey-top are exactly those for which $\neg\,\mathrm{Gal}(U, \kappa, 2^\kappa)$ holds.

The Galvin Property and Tukey Non-Top

A Galvin ultrafilter on κ is an ultrafilter satisfying $\mathrm{Gal}(U, \kappa, 2^\kappa)$; that is,

$$\forall \langle A_i : i < 2^\kappa \rangle \in [U]^{2^\kappa} \quad \exists I \in [2^\kappa]^\kappa \quad \bigcap_{i \in I} A_i \in U$$

Galvin's Theorem $\implies$ normal ultrafilters are Galvin $\implies$ not Tukey top.

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What guarantees Galvin ultrafilters?

Normality is not necessary: (Benhamou-Gitik 2022) showed that p-points are Galvin, and (Benhamou 2022+) showed that p-point limits of p-points are Galvin. Tukey analysis allows us to strengthen these results.

Lots of work on $(\neg)$ Galvin Property by Benhamou, Garti, Gitik, Poveda, Shelah.

Continuous Cofinal Maps

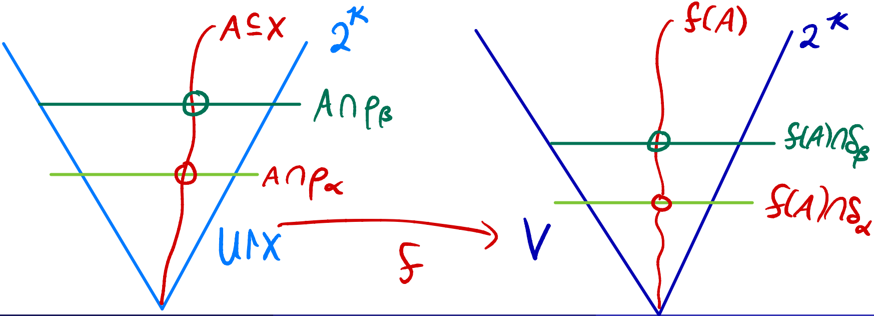
On ω, investigations of Tukey types of ultrafilters was opened up by the theorem (D.-Todorcevic 11) that p-points have continuous cofinal maps.

For κ measurable, we obtain a similar theorem, but the proof has some interesting differences and we actually obtain something a bit stronger.

In the following, consider 2^κ as a generalized Baire space with the topology generated by basic open sets of the form $\{x \in 2^\kappa : s \sqsubset x\}$, where $s \in 2^{<\kappa}$.

Theorem. (Benhamou-D.) Let κ be measurable, and suppose U is a p-point on κ and V is a uniform ultrafilter on κ such that $U \geq_T V$.

Then for each monotone cofinal map $f : U \to V$, there is an $X \in U$ such that the restriction of f to $U \restriction X$ is continuous and has image which is cofinal in V.



Continuous Cofinal Maps on 2^κ

More precisely, let $\pi : \kappa \to \kappa$ be the minimal non-constant function mod U ($[\pi]_U = \kappa$), and ρ be the function $\rho_\alpha = \sup(\pi^{-1}[\alpha + 1]) + 1$.

Then for any strictly increasing sequence $\langle \delta_\alpha \mid \alpha < \kappa \rangle$, there is an $X \in U$ such that for every $\alpha < \kappa$ and $A \in U \restriction X$, $f(A) \cap \delta_\alpha$ depends only on $A \cap \rho_\alpha$.

Moreover, there is a monotone function $\hat{f} : [\kappa]^{<\kappa} \to [\kappa]^{<\kappa}$ such that for each $A \in U \restriction X$, there are club many $\alpha < \kappa$ such that

$$\hat{f}(A \cap \rho_\alpha) = f(A) \cap \delta_\alpha$$

and in particular

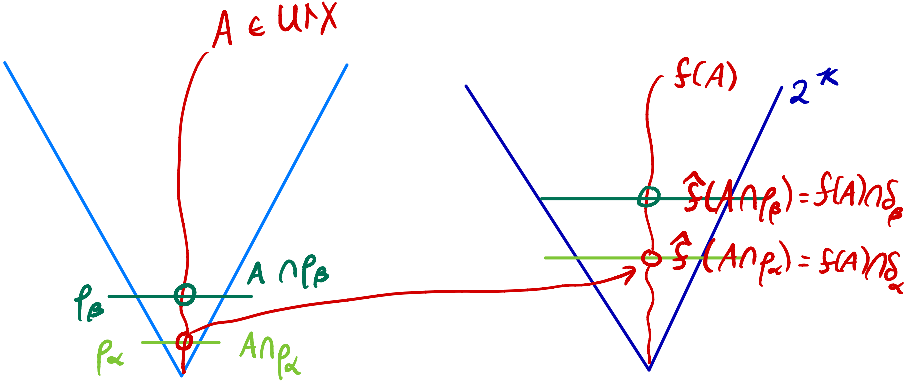
$$f(A) = \bigcup_{\alpha < \kappa} \hat{f}(A \cap \rho_\alpha)$$

Then letting $g(A) = \bigcup_{\alpha < \kappa} \hat{f}(A \cap X \cap \rho_\alpha)$, $g : \mathcal{P}(\kappa) \to \mathcal{P}(\kappa)$ is a continuous monotone map, $g \restriction U : U \to V$ is a cofinal map, and $g \restriction (U \restriction X) = f \restriction (U \restriction X)$.

Continuous Cofinal Maps on 2^κ

$\langle \rho_\alpha : \alpha < \kappa \rangle$ is determined by $\mathcal{U}$.

$\langle \delta_\alpha : \alpha < \kappa \rangle$ is arbitrary, fixed at beginning of proof.



Corollary for normal ultrafilters

If U is normal, then we obtain a stronger form of the above theorem.

Corollary. (B.-D.) If U is normal, then we have that $\rho_\alpha = \alpha + 1$ and thus $f(W) \cap \delta_\alpha = f((W \cap \alpha + 1) \cup (X \setminus \alpha + 1)) \cap \delta_\alpha$, for each $\alpha < \kappa$.

$\rho_\alpha = \sup(\pi^{-1}[\alpha + 1]) + 1.$

P-points have cofinal types of size 2^κ

Corollary. (B.-D.) There are only 2^κ many ultrafilters which are Tukey below a *p*-point. Moreover, every $\leq_T$-chain of *p*-points has order-type at most $(2^\kappa)^+$.

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Using the Hajnal free set lemma, we obtain:

Corollary. (B.-D.) If $\mathcal{X}$ is a set of more than $(2^\kappa)^+$-many p-points on κ, then there is a subset $\mathcal{Z} \in [\mathcal{X}]^{|\mathcal{X}|}$ such that every distinct $U, V \in Z$ are incomparable in the Tukey order.

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• In the Kunen-Paris model there are $2^{(\kappa^+)}$-many normal ultrafilters, $2^\kappa = \kappa^+$ and $2^{(\kappa^+)}$ can be made arbitrarily large. In particular, there are $2^{(2^\kappa)}$-many Tukey-incomparable normal ultrafilters.

Continuous cofinal maps for products of p-points

Theorem. (B.-D.) Suppose $U_1, \dots, U_n$ are *p*-points over $\kappa > \omega$, $\pi_1, \dots, \pi_n$ are functions such that $[\pi_i]_{U_i} = \kappa$, and V is a uniform ultrafilter on κ such that $U_1 \times \cdots \times U_n \geq_T V$.

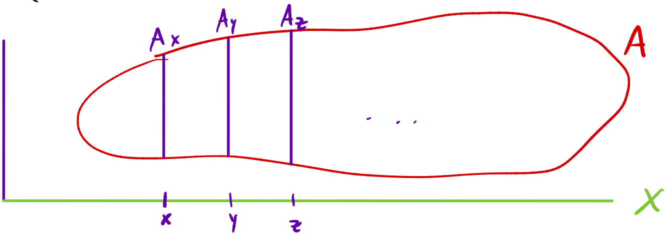
Then for each monotone cofinal map $f : U_1 \times \cdots \times U_n \to V$ and every sequence $\langle \delta_\alpha \mid \alpha < \kappa \rangle$, there are sets $X_i \in U_i$ such that the restriction of f to $U_1 \times \cdots \times U_n \restriction \langle X_1, \dots, X_n \rangle$ is monotone, cofinal, and uniformly continuous, i.e. for each $\alpha < \kappa$, $f(\langle B_1, \dots, B_n \rangle) \cap \delta_\alpha$ is determined by $\langle B_1 \cap \rho^1_\alpha, \dots, B_m \cap \rho^n_\alpha \rangle$.

This plus the next theorem on products are useful for finding the cofinal types in certain models.

Fubini Products

For any ultrafilter U over a set X and a sequence $\langle V_x \mid x \in X \rangle$ of ultrafilters over Y, define the *U-sum of the ultrafilters V_x*, to be the filter over $X \times Y$ defined as follows:

$$\sum_U V_x := \left\{ A \subseteq X \times Y : \{x \in X \mid \{y \in Y \mid \langle x, y \rangle \in B\} \in V_x\} \in U \right\}$$



If for every $x \in X$, $V_x = V$, we call the ultrafilter $U \cdot V := \sum_U V$ the *Fubini product of U and V*.

Products of Ultrafilters

Theorem. (D.-Todorcevic 11) If U is a rapid p-point on ω and V is any ultrafilter on ω, then $U \cdot V \equiv_T U \times V$.

However, there are non-rapid p-points for which $U <_T U \cdot U$.

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However, there are non-rapid p-points for which $U <_T U \cdot U$.

Theorem. (B.-D.) Let $\omega < \kappa$ be measurable. For every two κ-complete ultrafilters U, V over κ, $U \cdot V \equiv_T U \times V$.

Moreover, for $U_1, \dots, U_n$ any κ-complete ultrafilters over κ,

$$U_1 \cdot \ldots \cdot U_n \equiv_T \prod_{i=1}^{n} U_i$$

In particular, if U is a Galvin ultrafilter over a measurable cardinal κ (namely, $\mathrm{Gal}(U, \kappa, 2^\kappa)$) then for every n, U^n is Galvin.

Basic and basically generated ultrafilters

The notion of basic partial order comes from (Solecki-Todorcevic 04).

On ω, (D.-Todorcevic 11) showed that p-point = basic, and that basic implies not Tukey top. Then they showed that the relevant properties are preserved under Fubini iteration, and defined the notion of a basically generated ultrafilter.

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A κ-complete ultrafilter U over κ is called basic if for every sequence $\langle A_i \mid i < \kappa \rangle$ such that $\lim_{i \to \kappa} A_i = A \in U$ there is a subsequence $\langle A_{i_\alpha} \mid \alpha < \kappa \rangle$ such that $\cap_{\alpha < \kappa} A_{i_\alpha} \in U$.

(B.-D.) show that p-point on κ implies uniformly basic implies basic implies p-point. This yields a simple proof that p-points are Galvin.

Basic and basically generated ultrafilters

An ultrafilter U on κ is called basically generated if it is uniform, κ-complete, and has a κ-complete filter base $B \subseteq U$ with the property that to each sequence $\{A_i \mid i < \kappa\} \subseteq B$ converging to an element of B, there corresponds a function f such that for every $f \leq_{bd} g$, we have $\cap_{\alpha<\kappa} A_{g(\alpha)} \in U$.

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Basically generated implies not Tukey top.

Theorem. (B.-D.) Suppose that U and V_α, $\alpha < \kappa$, are basically generated ultrafilters on κ. Then $W := \sum_U V_\alpha$ is basically generated (with respect to the product topology on $\kappa \times \kappa$).

Open Problem

Is it consistent that there are ultrafilters over a measurable cardinal which are not Tukey-top and also not basically generated?

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Contrast with ultrafilters on ω:

- (D.-Todorcevic 11): stable ordered union ultrafilters are not Tukey top.
- (Blass-D.-Raghavan 15): the ultrafilter forced by $P(\omega^2)/\mathrm{fin}^{\otimes 2}$ is not Tukey top.
- (D. 16): the ultrafilters forced by $P(\omega^k)/\mathrm{fin}^{\otimes k}$ are all not Tukey top; moreover, they form an initial segment of the Tukey classes.

Models with various cofinal type structures

Corollary. Let U be a normal ultrafilter on a measurable cardinal κ. In $L[U]$:

1. There is no ultrafilter among the κ-complete ultrafilters over κ which is Tukey-top.
2. The κ-complete ultrafilters over κ form a single Tukey class which is the union of ω-many Rudin Keisler equivalence classes.

(2) follows from the facts that every κ complete ultrafilter in $L[U]$ is RK equivalent to U^n for some $n < \omega$; and we showed that $U^n \equiv_T U$.

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Contrast with ω: It is still open whether there is exactly one Tukey type for ultrafilters over ω in $L(\mathbb{R})[U]$, the Solovay model extended by a forced Ramsey ultrafilter U on ω.

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Theorem. (B.-D.)

1. Suppose that $U \lhd W$ are κ-complete ultrafilters and W is a p-point. Then $\neg(W \leq_T U)$.
2. If $U_1 \lhd U_2 \lhd \cdots \lhd U_n$ are normal ultrafilters, then $\neg(U_n \leq_T U_1 \times \cdots \times U_{n-1})$.
3. If $o(\kappa) = \alpha$ for $\alpha \leq \omega$, then there is a Tukey-chain of κ-complete ultrafilters of order type α.
4. Let $L[\vec{U}]$ be the Mitchell model for $o(\kappa) = \omega$ and let $\langle U_n \mid n < \omega \rangle$ be the $\lhd$-increasing sequence of ultrafilter on κ. Then the sequence $\langle [U_1 \times \cdots \times U_n]_T \mid n < \omega \rangle$ are strictly increasing, cofinal and unbounded in the Tukey order. In particular, there is no maximal Tukey class.

Models with various cofinal type structures

Propositions. (B.-D.)

1. Assume GCH and that κ is a measurable cardinal. Then Kunen-Paris construction of a model with many distinct normal ultrafilters provides model of GCH in which there is a Tukey-chain of normal ultrafilters on κ of order type $\omega + 1$.

2. For every $k < \omega$ it is consistent that $(P(k) \setminus \{\emptyset\}, \subseteq)$ can be embedded into the Tukey classes of κ-complete ultrafilter on κ.

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(Raghavan-Todorcevic 12): Ramsey ultrafilters on ω are Tukey minimal.

The issue with κ measurable is that there no Pudlák-Rödl analgogue for barriers on $[\kappa]^{<\kappa}$ because of Sierpiński's coloring.

Ultrapower Axiom

Theorem. (Goldberg)

1. Assume UA and that κ is a measurable cardinal with $o(\kappa) < 2^{(2^\kappa)}$. Then every κ-complete ultrafilter is Rudin-Keisler equivalent to a finite product of normal ultrafilters.
2. Under UA, the Mitchell order on normal ultrafilters is linear.

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Contrast with the wide array of consistent Tukey structures of ultrafilters on ω.

Thank you very much!

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[Benhamou-Dobrinen] *Cofinal types of ultrafilters over measurable cardinals*, arxiv:2304.07214