Infinite Structural Ramsey

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Ramsey Theory on ω

Theorem (Finite Ramsey Theorem)

Given $m < n$ and $2 \leq r$, there is a p large enough so that for any coloring $\chi : [p]^m \to r$, there is an $N \in [p]^n$ such that $\chi \restriction [N]^m$ is constant.

Theorem (Infinite Ramsey Theorem)

Given m, r and a coloring $\chi : [\omega]^m \to r$, there is an $N \in [\omega]^\omega$ such that $\chi \restriction [N]^m$ is constant.

Theorem (Galvin–Prikry)

Given a Borel coloring $c : [\omega]^\omega \to 2$, there is an $N \in [\omega]^\omega$ such that $\chi \restriction [N]^\omega$ is constant.

I. Structural Ramsey theory overview.

Structural Ramsey Theory

- (finite) Colorings of finite structures within finite structures.
  analogues of Finite Ramsey Theorem

- (finite-dimensional) Colorings of finite structures within infinite structures.
  analogues of Infinite Ramsey Theorem

- (infinite-dimensional) Colorings of infinite structures within infinite structures.
  analogues of Galvin-Prikry, Silver, or Ellentuck Theorems

Finite Structural Ramsey Theory

For structures $\mathbf{A}, \mathbf{B}$, write $\mathbf{A} \leq \mathbf{B}$ iff $\mathbf{A}$ embeds into $\mathbf{B}$.

$\binom{\mathbf{B}}{\mathbf{A}}$ denotes the set of all copies of $\mathbf{A}$ in $\mathbf{B}$.

A class $\mathcal{K}$ of finite structures has the **Ramsey Property** if given $\mathbf{A} \leq \mathbf{B}$ in $\mathcal{K}$ and r, there is $\mathbf{C} \in \mathcal{K}$ so that

$$\forall \chi : \binom{\mathbf{C}}{\mathbf{A}} \to r \quad \exists \mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}},\ \chi \restriction \binom{\mathbf{B}'}{\mathbf{A}} \text{ is constant.}$$

Lots of work done! (e.g., Nešetřil–Rödl, Hubička–Nešetřil)

Examples. Finite linear orders, finite ordered graphs, finite ordered k-partite graphs, finite tournaments, finite ordered k-regular hypergraphs, finite ordered 3-regular hypergraphs omitting pyramids, finite Boolean algebras,...

Finite-dimensional Structural Ramsey Theory

Let $\mathcal{K}$ be a Fraïssé class of finite structures with limit $\mathbf{K}$.

$\mathbf{K}$ has **finite big Ramsey degrees** if for each finite $\mathbf{A} \in \mathcal{K}$, $\exists t$ s.t. $\forall r$,

$$\forall\ \chi : \binom{\mathbf{K}}{\mathbf{A}} \to r,\ \exists \mathbf{K}' \in \binom{\mathbf{K}}{\mathbf{K}} \text{ such that } \left|\chi \restriction \binom{\mathbf{K}'}{\mathbf{A}}\right| \leq t.$$

The **big Ramsey degree** of $\mathbf{A}$ in $\mathbf{K}$ = $\mathrm{BRD}(\mathbf{A}, \mathbf{K}) = \mathrm{BRD}(\mathbf{A})$ is the least such t, if it exists.

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- $\mathrm{BRD}(\mathbf{A}) = 1$ means exact analogue of Infinite Ramsey Theorem.
- (Hjorth 2008) If $|\mathrm{Aut}(\mathbf{K})| > 1$, then $\mathcal{K}$ has some BRD > 1.

BRD's are really about the optimal structural expansions for which Ramsey's Theorem holds. (canonical partitions)

Theorem (Kechris–Pestov–Todorcevic, 2005)

A Fraïssé class $\mathcal{K}$ of finite structures has the Ramsey property if and only if $Aut(\mathbf{K})$ is extremely amenable, where $\mathbf{K}$ is the homogeneous structure universal for $\mathcal{K}$.

Topological Dynamics and Ramsey Theory

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Questions (KPT).

(1) Which homogeneous structures have big Ramsey degrees?

(2) What topological dynamics correspond to big Ramsey degrees?

(3) Which homogeneous structures have ∞-dim'l Ramsey Theory?

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Theorem (Zucker, 2019)

If $\mathbf{K}$ has a big Ramsey structure, then $Aut(\mathbf{K})$ admits a unique universal completion flow.

A subset $\mathcal{X} \subseteq [\omega]^\omega$ is **Ramsey** if each for $M \in [\omega]^\omega$, there is an $N \in [M]^\omega$ such that $[N]^\omega \subseteq \mathcal{X}$ or $[N]^\omega \cap \mathcal{X} = \emptyset$.

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Axiom of Choice $\Longrightarrow \exists \mathcal{X} \subseteq [\omega]^\omega$ which is not Ramsey.

Solution: restrict to topologically 'definable' sets.

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Galvin–Prikry. Borel sets are (completely) Ramsey.

Ellentuck. A set is (completely) Ramsey iff it has the property of Baire in the Vietoris (=Ellentuck) topology.

Topological Ramsey spaces: Points are infinite sequences, topology is induced by finite heads and infinite tails, and
every subset with the property of Baire is completely Ramsey.
(Carlson–Simpson 1990; Todorcevic 2010.)

Infinite-dimensional structural Ramsey Theory

We assume the universe of $\mathbf{K}$ is ω so that $\binom{\mathbf{K}}{\mathbf{K}}$ is a subspace of $[\omega]^\omega$ with the metric topology.

$\mathbf{K}$ has an infinite-dimensional Ramsey Theorem iff there is a subspace $\mathcal{S}$ of $\binom{\mathbf{K}}{\mathbf{K}}$ which satisfies a Galvin-Prikry, Silver, or Ellentuck-style theorem.

Big Ramsey degrees pose a constraint.

Question (KPT). Develop infinite-dimensional Ramsey theory for $(\mathbb{Q}, <)$, the Rado graph, ...

II. Prototype Example: Ramsey theory of $(\mathbb{Q}, <)$

- D. Devlin's Theorem.
- Retrospective of D. Devlin's Theorem using Coding Trees of 1-Types.
- Infinite-dimensional Ramsey theory via topological characterizations.

Sierpiński's Unavoidable 2-Coloring

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Key Idea: Enumerate $\mathbb{Q}$ as $\langle r_0, r_1, r_2, \dots \rangle$

$$\text{Define a coloring : for } i < j,\ c(\{r_i, r_j\}) = \begin{cases} \text{red} & \text{if } r_i < r_j \\ \text{blue} & \text{if } r_j < r_i \end{cases}$$

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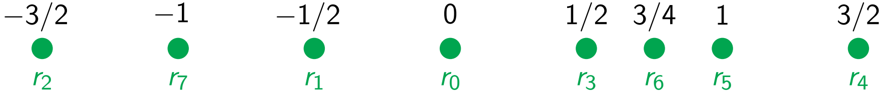

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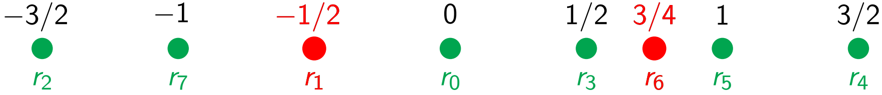

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$-3/2$	-1	$-1/2$	0	$1/2$	$3/4$	1	$3/2$
r_2	r_7	r_1	r_0	r_3	r_6	r_5	r_4

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Coloring Finite Sets of Rationals

Theorem (D. Devlin, 1979)

Given m, if the m-element subsets of $\mathbb{Q}$ are colored by finitely many colors, then there is a subcopy $\mathbb{Q}' \subseteq \mathbb{Q}$ forming a dense linear order in which the m-element subsets take no more than $C_{2m-1}(2m-1)!$ colors. This bound is optimal.

m	**Bound**
1	1
2	2
3	16
4	272

C_i is from

$\tan(x) = \sum_{i=0}^{\infty} C_i x^i$

- Galvin (1968) The bound for pairs is two.
- Laver (1969) Upper bounds for all finite sets.

Well-order $\mathbb{Q}$ as $\langle r_i : i < \omega \rangle$. This induces the tree of quantifier-free complete 1-types. c_n denotes the 1-type of r_n over $\{r_i : i < n\}$.

Retrospective of Devlin's Theorem via Coding Trees

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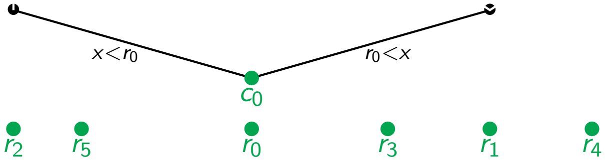

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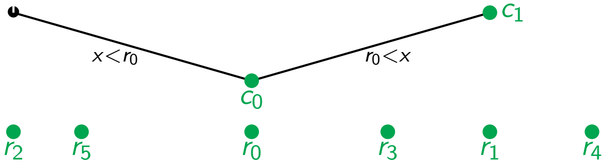

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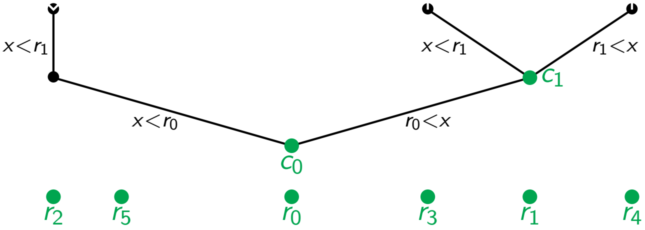

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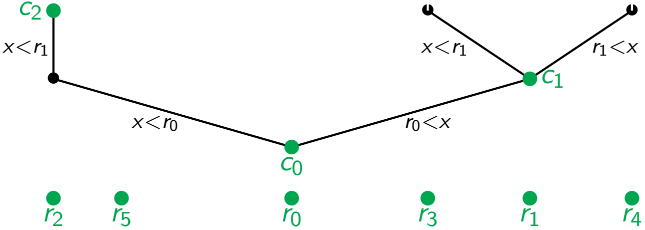

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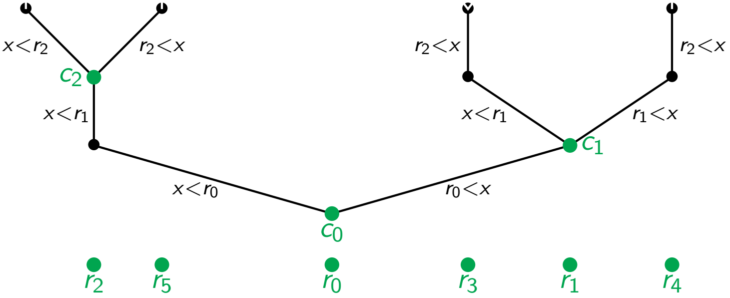

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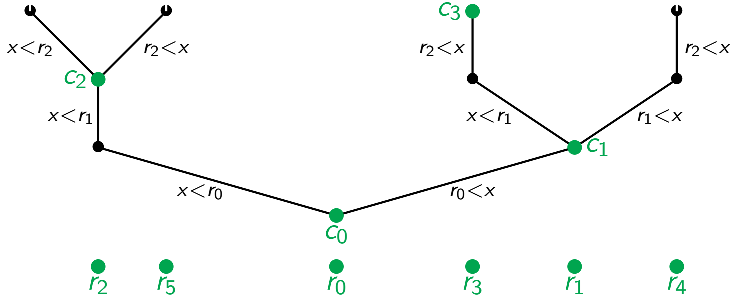

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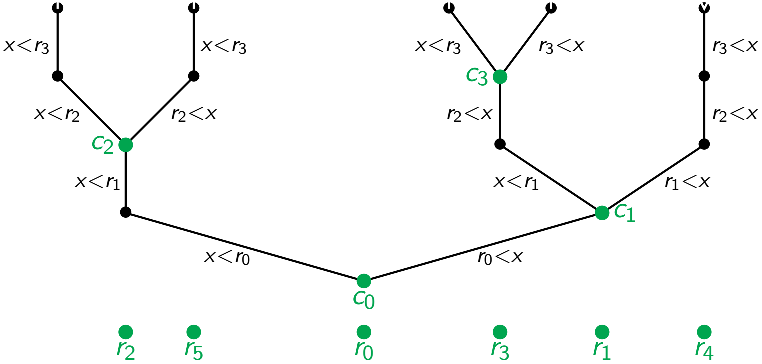

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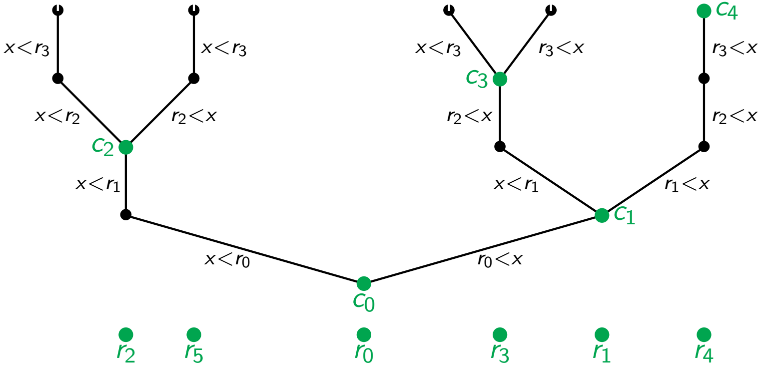

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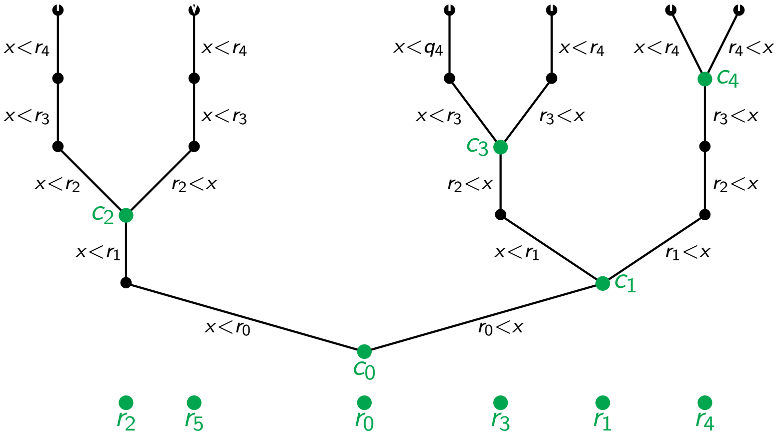

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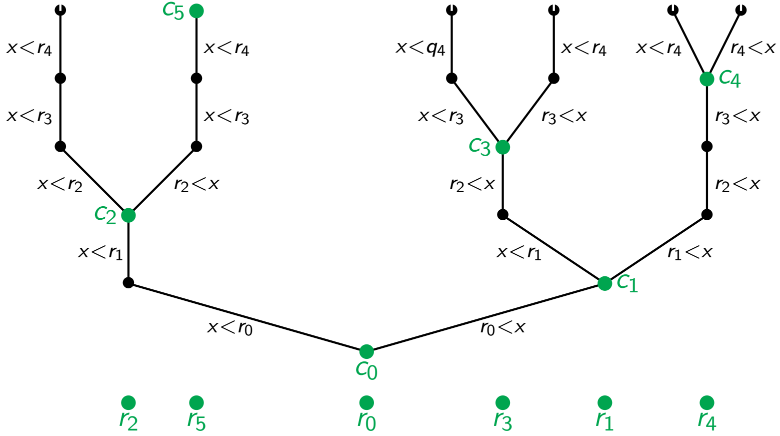

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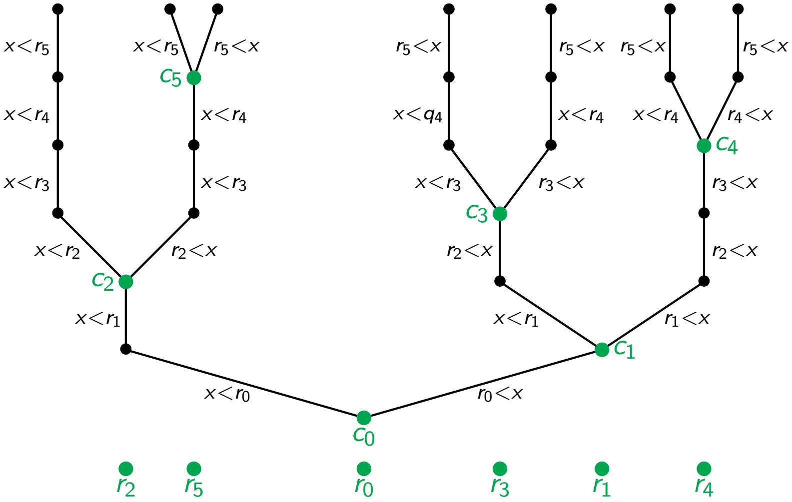

Devlin's Diagonal Antichain in a Coding Tree

An antichain is **diagonal** if any two nodes in its meet closure have different lengths, and all splitting nodes have branching degree 2.

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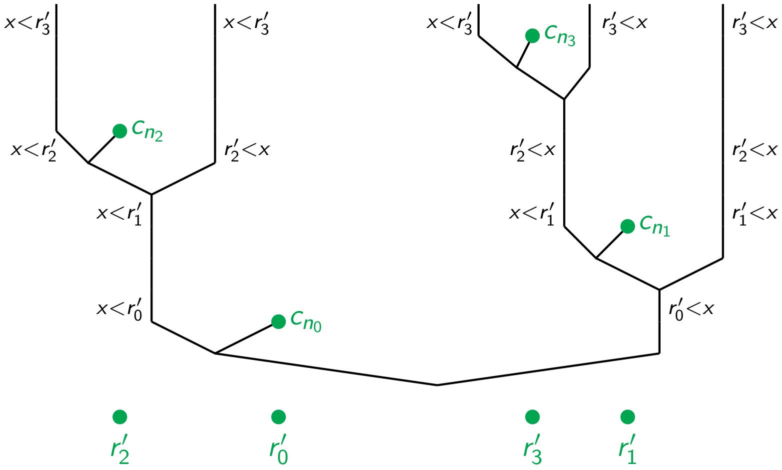

Exact Big Ramsey Degrees in $\mathbb{Q}$

The big Ramsey degree of $(m, <)$ in $\mathbb{Q}$ is the number of isomorphism classes of diagonal antichains with m terminal nodes.

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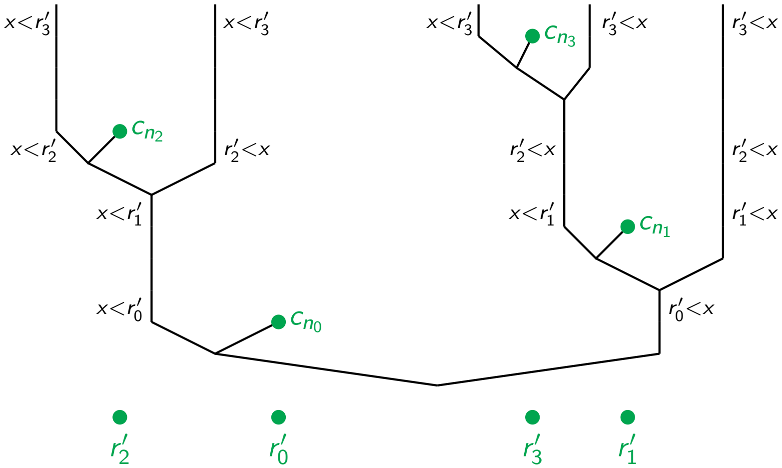

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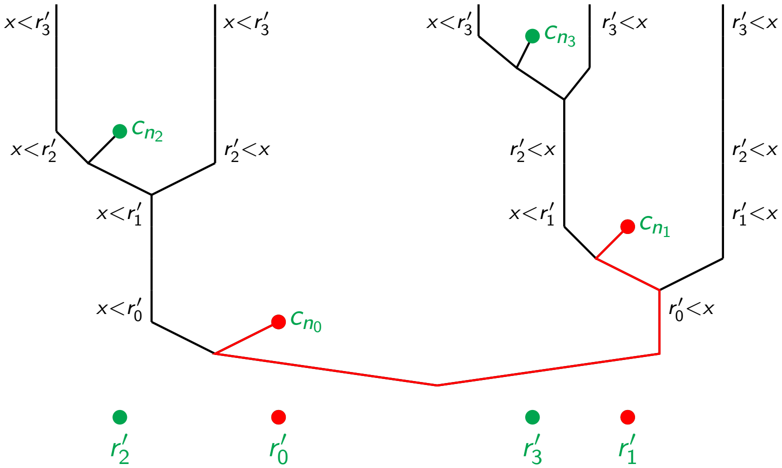

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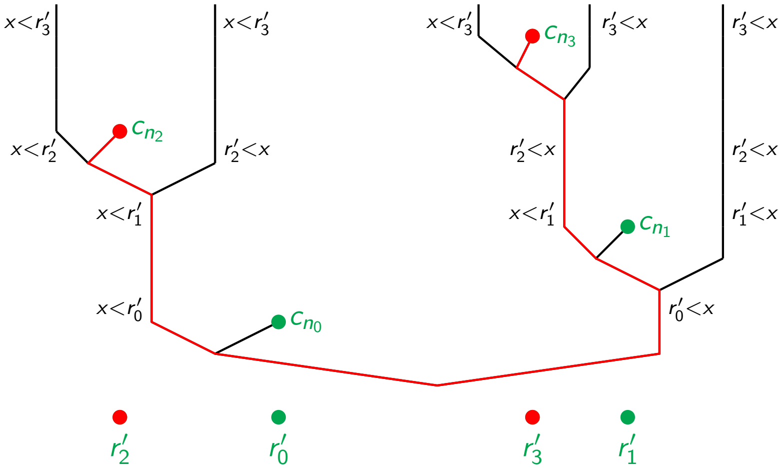

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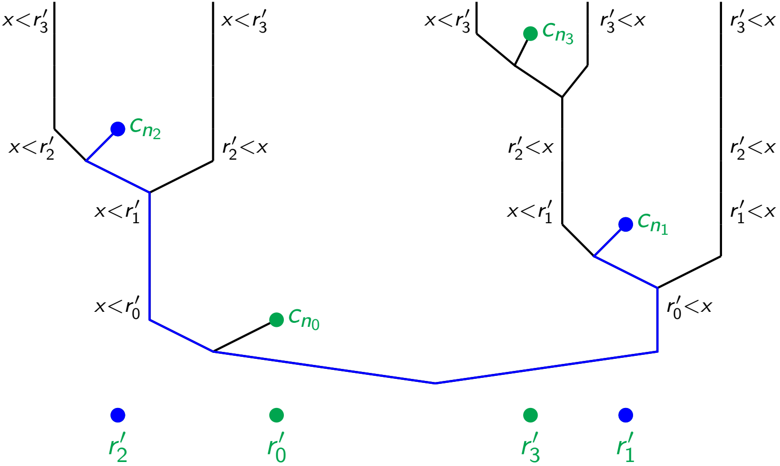

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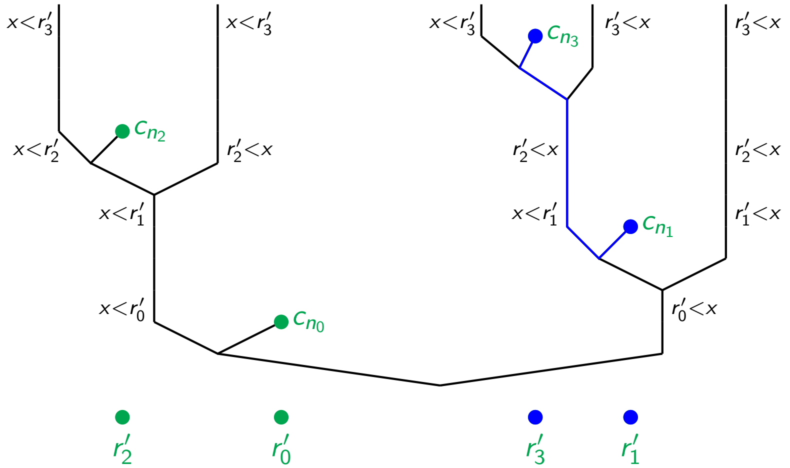
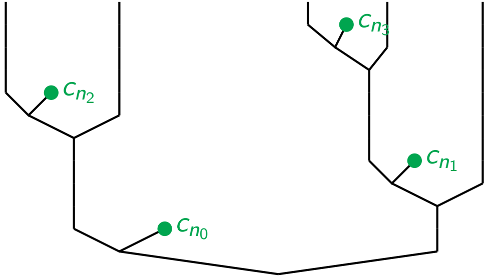


Infinite-Dimensional Ramsey Theory for $(\mathbb{Q}, <)$

Theorem (D., 2019)

Let Δ be a diagonal antichain representing $\mathbb{Q}$. Let $\mathcal{R}(\Delta)$ be the collection of all subsets of Δ which are isomorphic to Δ. Then $\mathcal{R}(\Delta) \subseteq \binom{\mathbb{Q}}{\mathbb{Q}} \subseteq [\omega]^\omega$ is a **topological Ramsey space***: A subset of $\mathcal{R}(\Delta)$ is Ramsey iff it has the property of Baire with respect to the Ellentuck topology.*



III. Catalogue of results in big Ramsey degrees and infinite-dimensional structural Ramsey theory.

Major Big Ramsey Degree results $\leq$ 2010

- 1933. BRD(Pairs, $\mathbb{Q}$) ≥ 2. (Sierpiński)
- 1975. BRD(Edge, $\mathcal{R}$) ≥ 2. (Erdős, Hajnal, Pósa)
- 1979. $(\mathbb{Q}, <)$: All BRD computed. (D. Devlin)
- 1986. BRD(Vertex, $\mathcal{H}_3$) $= 1$. (Komjáth, Rödl)
- 1989. BRD(Vertex, $\mathcal{H}_n$) $= 1$. (El-Zahar, Sauer)
- 1996. BRD(Edge, $\mathcal{R}$) $= 2$. (Pouzet, Sauer)
- 1998. BRD(Edge, $\mathcal{H}_3$) $= 2$. (Sauer)
- 2006, 2008. The Rado graph: All BRD characterized; computed. (Laflamme, Sauer, Vuksanović); (J. Larson)
- 2008. Ultrametric spaces: BRD computed. (Nguyen Van Thé)

Major Big Ramsey Degree results $\leq$ 2010

- 2009. BRD of κ-dense linear order, κ measurable. (Džamonja, Larson, Mitchell)
- 2009. BRD of κ-Rado graph, κ measurable. (Džamonja, Larson, Mitchell)
- 2010. Dense Local Order $\mathbf{S}(2)$: All BRD computed. Also $\mathbb{Q}_n$. (Laflamme, Nguyen Van Thé, Sauer)
- 2010. Weak indivisibility of homogenous metric spaces (Nguyen Van Thé, Sauer)
- 2019. FBRD of profinite graphs. (Huber, Geschke, Kojman, 2008+)

BRD and ∞-dim'l Results > 2010 (arxiv dates)

- 2017. FBRD for Triangle-free Henson graph: Close Upper Bounds. (D., 2020 JML) Coding Trees and Forcing

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- 2018. FBRD for certain homogeneous metric spaces and certain universal structures. (Mašulović, 2020 J. Comb. Th. Ser. A) Category Theory

BRD and ∞-dim'l Results 2019 (arxiv dates)

- FBRD for all Henson graphs: Good Upper Bounds. (D., 2023 JML) Coding Trees and Forcing
- ∞-dimensional RT for Borel sets of Rado graphs. (D., 202* Prague Ramsey Theory DocCourse) Coding Trees and Forcing
- FBRD for Countable ordinals. (Mašulović, Šobot, 2021 Combinatorica) Induction
- FBRD for 3-uniform hypergraphs. (Balko, Chodounský, Hubička, Konečný, Vena, 2022 Combinatorica) Product Milliken Theorem
- Expository article on "Ramsey Theory on Infinite Structures and the Method of Strong Coding Trees", (D., 2020 Contemporary Logic and Computing)
- Big Ramsey spectra of countable scattered chains. (Mašulović, 2023 Order)

BRD and ∞-dim'l Results 2020 (arxiv dates)

- Exact BRD computed for all circular directed graphs $\mathbb{S}(n)$. (Dasilva Barbosa) Category Theory
- FBRD for binary rel. FAP classes. (Zucker, 2022 Adv. Math.) Coding Trees and Forcing
- Expository paper on BRD problems for undergraduates. (D., Gasarch, 2020 ACM SIGACT News)
- Computability theory of Halpern-Läuchli, Milliken, $\mathbb{Q}$, Rado. (Anglès d'Auriac, Cholak, Dzhafarov, Monin, Patey, 2023 AMS Memoirs)
- FBRD for Homogeneous partial order. (Hubička) RT for parameter words. First non-forcing proof for $\mathbf{H}_3$.
- Exact BRD for binary SDAP$^+$ and indivisibility for higher arity SDAP$^+$ structures. (Coulson, D., Patel) (2 papers) Coding Trees and Forcing. Consolidates many previous theorems. First proof with no envelopes.
- BRD in universal inverse limit structures. (D., Wang, 2023 AFML) Halpern-Läuchli, Blass Thm.

BRD and ∞-dim'l Results 2021 and 2022 (arxiv dates)

- 2021. FBRD for Homogenous graphs with forbidden cycles (metric spaces). (Balko, Chodounský, Hubička, Konečný, Nešetřil, Vena) RT for parameter words.
- 2021. Expository/survey article (D., Proceedings of the 2022 ICM)
- 2021. Exact BRD for binary rel. FAP. (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker, JEMS, to appear) Coding Trees
- 2022. ∞-dimensional RT structures with SDAP$^+$. (recovers Exact BRD). (D.) Coding Trees and Forcing. First top. Ramsey spaces of Fraïssé structures.
- 2022. Big Ramsey degrees in ultraproducts of finite structures. (Bartošová, Džamonja, Patel, Scow, APAL 2024) Ultraproducts
- 2022. Carlson-Simpson in ACA_0^+. (Anglès d'Auriac, Mignoty, Patey, 2023 APAL) Hence, FBRD of generic partial order and of triangle-free Henson graph provable in ACA_0^+.

BRD and ∞-dim'l Results 2023 (arxiv dates)

- BRD and infinite languages. (Braunfeld, Chodounský, de Rancourt, Hubička, Kawach, Konečný, Adv. Combinatorics, to appear)
  Laver's Product Milliken Theorem for infinitely many trees
- Exact BRD. Homogeneous partial order. (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker, TAMS, to appear*)
  Hubička's Carl.-Simp. methods
- Type-respecting amalgamation and big Ramsey degrees. (Aranda, Braunfeld, Chodounský, Hubička, Konečný, Nešetřil, Zucker, 2023 EUROCOMB Abstracts)
- BRD in the metric setting. (Bice, de Rancourt, Hubička, Konečný)
- Countable chains with finite big Ramsey spectra. (Dasilva Barbosa, Mašulović, Nenadov, 2024 Order)
- Connections between big and small Ramsey degrees. (Mašulović)

BRD and ∞-dim'l Results 2023 (arxiv dates)

- ∞-dimensional RT for binary rel. FAP. (recovers Exact BRD). (D., Zucker) Coding Trees and Forcing
- Big Ramsey degrees of Countable Ordinals. (Boyland, Gasarch, Hurtig, Rust) elementary methods
- Characterizing countable chains with finite big Ramsey spectra. (Dasilva Barbosa, Mašulović, Nenadov)
- Box Ramsey and Canonical Colourings. (Dasilva Barbosa)
- Ramsey theorem for trees with successor operation. (Balko, Chodounský, D., Hubička, Konečný, Nešetřil, Vena, Zucker) Combinatorial Forcing. Consolidates many FBRD results and Milliken and Carlson-Simpson Theorems in one framework.
- Indivisibility of $\mathbf{H}_3$ and computability. (Gill)

BRD and ∞-dim'l Results 2024 and 2025

- On Ramsey degrees, compactness and approximability. (Mašulović)
- Dual Ramsey degrees. (Džuklevski, Mašulović)
- Survey on big Ramsey structures. (Hubička, Zucker)
- BRD of structures with finite monomorphic decomposition. (Toljić, Mašulović)
- Pseudotree is indivisible but has antichains with ∞ BRD. (Chodounský, Eskew, Weinert)
- Finite chains in pseudotree have FBRD. (Chodounský, D., Weinert) Coding trees and forcing
- FBRD of triangle-free Henson graph implies ACA_0'. (Cholak, D., McCoy) Coding trees
- Forcing methods in the arithmetical realm. (Cholak, D., Towsner) Coding trees
- More ongoing work...

IV. Methods and Results.

1. Halpern-Läuchli and Milliken Theorems and classic methods
2. Coding Trees and Forcing Methods for ZFC results
3. Characterizations of BRD's
4. Infinite-dimensional structural Ramsey theory
5. More techniques: Category Theory, Carlson-Simpson, $\mathcal{S}$-trees, ...
6. Structures with small Ramsey degrees but infinite big Ramsey degrees
7. Arithmetical Results

Halpern-Läuchli Ramsey Theorem on Trees

Notation:

$$\bigotimes_{i<d} T_i := \bigcup_{n<\omega} \prod_{i<d} T_i(n)$$

Theorem (Halpern-Läuchli, 1966)

Let $T_i \subseteq \omega^{<\omega}$, $i < d$, be finitely branching trees with no terminal nodes. Given a coloring $\chi : \bigotimes_{i<d} T_i \to 2$, there are strong subtrees $S_i \leq T_i$ with nodes of the same lengths such that χ is constant on $\bigotimes_{i<d} S_i$.

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Remarks.

- HL was distilled as a key lemma in the proof that the Boolean Prime Ideal Theorem is strictly weaker than the Axiom of Choice over ZF. (Halpern-Lévy, 1971)

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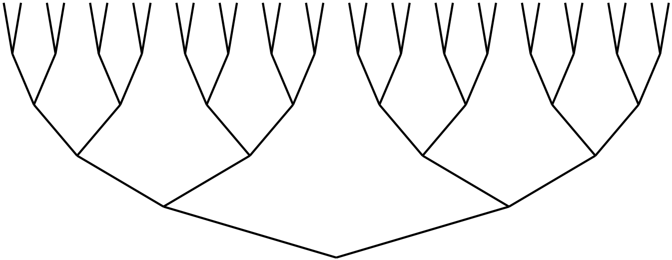
Remarks.

- HL was distilled as a key lemma in the proof that the Boolean Prime Ideal Theorem is strictly weaker than the Axiom of Choice over ZF. (Halpern-Lévy, 1971)
- Harrington gave a proof using the mechanism of forcing to do unbounded searches for finite objects.

Milliken's Theorem

Theorem (Milliken, 1979)

Let $T \subseteq \omega^{<\omega}$ be a finitely branching tree with no terminal nodes. Given $n \geq 1$ and a coloring of all n-strong subtrees of T into finitely many colors, there is an infinite strong subtree of T in which all n-strong subtrees have the same color.
There is also a product version for more than one tree.



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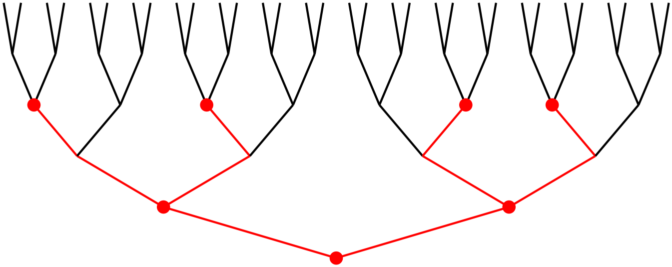

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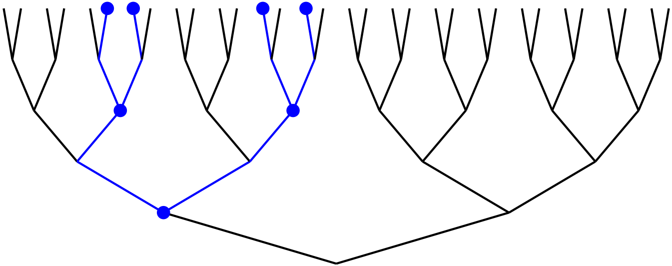

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Remark.

- Halpern–Läuchli Theorem forms the pigeonhole principle in the proof of Milliken Theorem.

Classic methodology: [Laflamme, Sauer, Vuksanovic, 2006]

1. Tree $k^{<\omega}$ represents a universal structure for Age($\mathbf{K}$) where $\mathbf{K}$ is an unrestricted homogeneous structure with n binary relations, where $k = 2^n$. (e.g., Rado graph, generic digraph, finite superpositions)

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- This methodology works for $\mathbb{Q}$ (see Todorcevic book), for unrestricted structures with binary relations (Laflamme–Sauer–Vuksanovic), and for $\mathbb{Q}_n$ with new Milliken Thm (Laflamme–Nguyen Van Thé–Sauer).

Coding Trees of 1-Types

Let $\mathbf{K}$ be a homogeneous structure with enumerated vertices $\langle v_i : i < \omega\rangle$.

Let $\mathbf{K}_n = \mathbf{K} \restriction \{v_i : i < n\}$.

This **coding tree of 1-types** $\mathbb{S}(\mathbf{K})$ is the set of all (quantifier-free) complete 1-types over $\mathbf{K}_n$, $n < \omega$. c_n distinguishes the node that is the 1-type of v_n over $\mathbf{K}_n$. The tree-ordering is inclusion.

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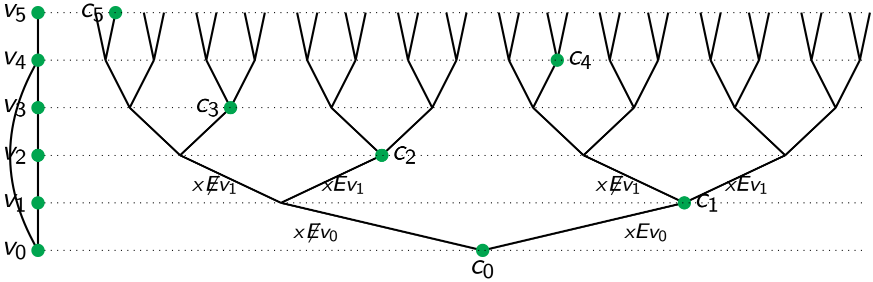
- Work directly on homogeneous structures, rather than just universals.
- Useful for forbidden substructures and for ∞-dim'l Ramsey Theory.

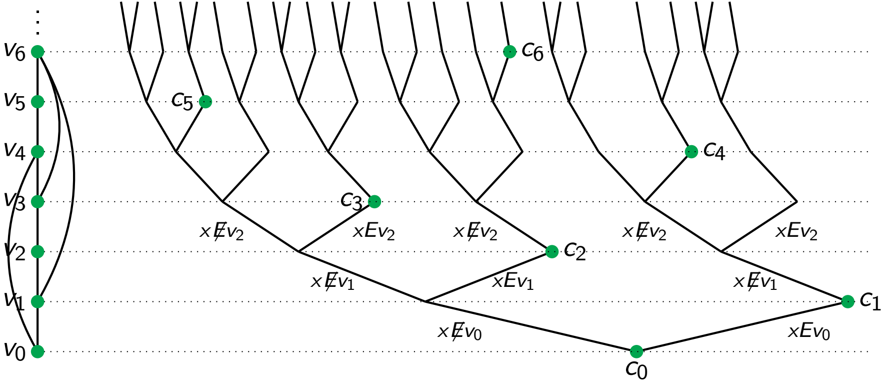
Coding trees were first developed in my work on the triangle-free Henson graph. The above definition appeared in Coulson, D., Patel in 2020.

Coding Tree of 1-types for the Rado graph



An Optimal Coding Tree for $\mathbf{H}_3$

Enumerate $\mathbf{H}_3$ to reduce number of diaries.



Forcing Ramsey Theorems on Coding Trees

- Let $\mathbb{S}$ be a coding tree of 1-types for an enumerated structure $\mathbf{K}$.
- Fix a finite subtree $A \subseteq \mathbb{S}$ and fix a level set X in $\mathbb{S}$ end-extending A.
- Color all copies of X extending A into two colors.
- Build a subtree $T \subseteq \mathbb{S}$ representing a subcopy of $\mathbf{K}$ in which all copies of X extending A have the same color.

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Three challenges:
1) Figure out the right partial order to force with.
2) Find good starting nodes for building a subtree.
3) Build a subtree encoding $\mathbf{K}$ in which all copies of X have the same color.

All of this is done in ZFC or within some ground model.

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- Then prove a Milliken-style theorem using induction.

These methods were developed in 2015–2017 for the Henson graph.

Results using Coding Trees and Forcing Methods

Theorem (D., JML 2020 and 2023)

The triangle-free and more generally all k-clique-free Henson graphs $\mathbf{H}_k$ have finite big Ramsey degrees.

Proofs directly reproduce indivisibility. By starting with certain coding trees, can reduce upper bounds, or even produce exact ones (later).

A small tweak of the trees in [D.2020] produces exact big Ramsey degrees.

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The triangle-free and more generally all k-clique-free Henson graphs $\mathbf{H}_k$ *have finite big Ramsey degrees.*

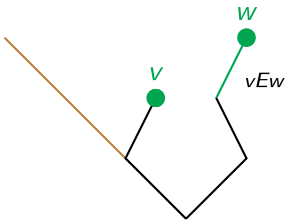
Theorem (D. and independently, Balko, Chodounský, Hubička, Konečný, Vena, Zucker, 2020)

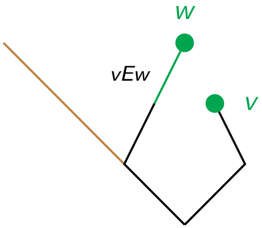
Exact big Ramsey degrees of the triangle-free Henson graph $\mathbf{H}_3$ *are characterized by*

1. *Diagonal antichains;*
2. *First levels of pairs coding of edges with a common vertex in* $\mathcal{H}_3$*;*
3. *Controlled coding levels;*
4. *First level off of leftmost branch.*

$T(\text{Edge}, \mathcal{H}_3) = 2$

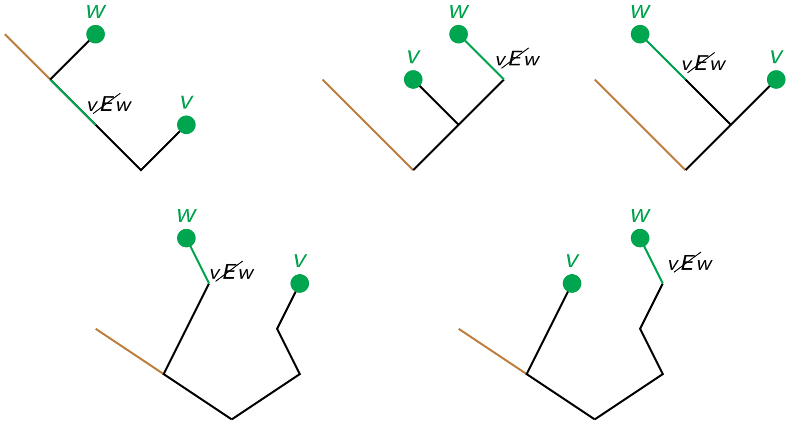
These are the unavoidable patterns representing edges.





$T(\text{Non-Edge}, \mathcal{H}_3) = 5$

These are the unavoidable patterns representing non-edges.



Results using Coding Trees and Forcing Methods

Fix a language $\mathcal{L}$ with finitely many relations of arity at most 2.

An $\mathcal{L}$-structure is **irreducible** if any two vertices are in some relation: e.g., finite clique, finite tournament, triangle with 2 red edges and one blue edge.

Free amalgamation classes are exactly of the form $\mathrm{Forb}(\mathcal{F})$, where $\mathcal{F}$ is a set of finite **irreducible** structures.

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Theorem (Zucker, Adv. Math. 2022)

All finitely constrained binary FAP classes have finite big Ramsey degrees.

Proof uses coding trees and forcing methods.

BRD of Finitely Constrained Binary FAP Classes

Theorem (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker, JEMS 2025+)

The exact big Ramsey degrees of finitely constrained binary FAP classes are characterized by the following:

1. *Diagonal antichains*
2. *Controlled splitting levels*
3. *Controlled age-change levels (essential changes in the class of structures which can be glued above a finite structure to make a member of $\mathcal{K}$)*
4. *Controlled coding levels (reducing the ages of the extending class as much as possible)*
5. *Controlled paths (only matter for non-trivial unary relations)*

Results using Coding Trees and Forcing Methods

An unexpected application of coding trees:

Theorem (Coulson–D.–Patel 2020+)

Let $\mathcal{L}$ be a finite relational language and let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit satisfying the SDAP$^+$. Let $\mathbf{K} = Flim(\mathcal{K})$.

I. $\mathbf{K}$ is indivisible: vertices have big Ramsey degree 1.

II. If $\mathcal{L}$ has no relations of arity greater than two, then $\mathbf{K}$ has big Ramsey degrees characterized by diagonal antichains.

This class of structures includes

- $\mathbb{Q}$, $\mathbb{Q}_n$ [Laflamme, Nguyen Van Thé, Sauer], $\mathbb{Q}_{\mathbb{Q}}$, $(\mathbb{Q}_{\mathbb{Q}})_n$,
- Rado graph, all structures in [LSV], generic k-partite graph, ordered versions of these.

∞-dim'l structural RT via coding trees and forcing

Theorem (D. 2019+)

The Rado graph has versions of the Galvin–Prikry Theorem.

Theorem (D. 2022+)

Fraïssé structures satisfying SDAP$^+$ with finitely many relations of arity at most two have analogues of the Galvin–Prikry Theorem which directly recover big Ramsey degrees. Moreover, if $\mathbf{K}$ has a certain amount of rigidity, we obtain analogues of Ellentuck's Theorem.

Theorem (D., Zucker 2023+)

Fix a finitely constrained binary free amalgamation class $\mathcal{K}$ and let $\mathbf{K} =$ Flim$(\mathcal{K})$. Then $\mathbf{K}$ has infinite-dimensional Ramsey theory which directly recovers exact big Ramsey degrees in (BCDHKVZ 2021).

Abstract Ramsey Theorem from weaker system

Theorem (Todorcevic)

Suppose that $(\mathcal{R}, \mathcal{S}, \leq, \leq_{\mathcal{R}})$ with finite restriction maps satisfying axioms **A.1–A.4**, *and that $\mathcal{S}$ is closed. Then the field of $\mathcal{S}$-Ramsey subsets of $\mathcal{R}$ is closed under the Souslin operation and it coincides with the field of $\mathcal{S}$-Baire subsets of $\mathcal{R}$.*

When $\mathcal{R} = \mathcal{S}$, this theorem implies the Abstract Ellentuck Theorem.

Theorem (D., Zucker 2023+)

The conclusion of the above theorem still holds when axiom **A.3(2)** *is replaced by the weaker existence of an* **A.3(2)**-*ideal.*

Coding trees of 1-types = directly working on homogeneous structures. They are useful in obtaining exact bounds, infinite-dimensional Ramsey theory of homogeneous structures, and finding tight computable bounds.

Other methods can give shorter proofs of upper bounds and can give some results that forcing proofs so far cannot. But they work on universal structures so cannot give eact upper bounds nor infinite-dimensional Ramsey theorems directly recovering exact BRD.

Non-forcing techniques

• Category Theory. Developed by Mašulović and Dasilva Barbosa to get transfer principles for big Ramsey degrees. Lots of work!

• Hubička: Carlson–Simpson and related theorems for the generic partial order. Robust method for metric spaces, and even $\mathbf{H}_3$.

Theorem (Hubička 2020+)

The generic partial order with linear extension has finite big Ramsey degrees. And a short combinatorial proof of $\mathrm{FBRD}(\mathbf{H}_3)$.

Theorem (Balko, Chodounský, D., Hubička, Konečný, Vena, Zucker, TAMS? 2025+)

Exact big Ramsey degrees of the generic partial order with linear extension.

A generalizing Ramsey Theorem

Theorem (Balko, Chodounský, D., Hubička, Konečný, Nešetřil, Zucker, 2023+)

A Ramsey theorem which is a common generalization of Carlson–Simpson Theorem and Milliken Tree Theorem for regularly branching trees.

Gives new shape-preserving Tree Ramsey Theorem.

Gives a non-forcing proof of Zucker's Theorem of finite big Ramsey degrees of finitely constrained binary relational FAP structures.

Unrestricted Structures in Infinite Languages

Theorem (Braunfeld, Chodounský, de Rancourt, Hubička, Kawach, Konečný, Adv. Combinatorics 2024)

Every unrestricted homogeneous structure with finitely many relations of any arity (and possibly infinitely many relations) has finite big Ramsey degrees.

If an unrestricted homogeneous structure has infinitely many relations with the same arity, then it has infinite big Ramsey degrees.

Proof uses a version of Laver's Ramsey Tree Theorem for level set products on infinitely many trees.

Pseudotrees: Some finite, some infinite BRD

A **dendrite** is a locally connected continuum that contains no simple closed curve. The **order** of a point x is the number of connected components obtained after removing x. A **ramification point** is a point of order ≥ 3.

For $P \subseteq \{3, 4, \ldots, \omega\}$, a generalized **Ważewski dendrite $\mathbf{W_P}$** is a dendrite such that each ramification point has order in P, and each arc in W_P contains a ramification point.

- W_P is unique up to homeomorphism. (Charatonik, Dilks)

Theorem (Kwiatkowska, JSL 2018)

If $P \subseteq \{3, 4, \ldots, \omega\}$ is finite, then the universal minimal flow of the homeomorphism group $H(W_P)$ is metrizable, and is computed explicitly. If P is infinite, then the UMF of $H(W_P)$ is not metrizable.

Dendrites $\longrightarrow$ Unrooted Pseudotrees

M_P = the set of all ramification points of W_P, a countable structure.

The pseudotree Ψ_P is the rooted version of M_P.

Theorem (Kwiatkowska, JSL 2018)

$\mathrm{Age}(\Psi_P)$ *has a precompact Ramsey expansion.*

A **pseudotree** is a rooted structure $\Psi = \langle T; \prec, \wedge, <_{\mathrm{lex}} \rangle$ such that for each $t \in T$, $\{s \in T : s \prec t\}$ is a linear order, $s \wedge t$ is the **meet** of s and t, and $<_{\mathrm{lex}}$ is the lexicographic order between $\prec$-incomparable nodes.

Ψ denotes the 2-branching pseudotree.

Ψ is homogeneous for the Fraïssé-HP class of finite 2-branching rooted trees.

2-branching Pseudotree

Theorem (Chodounský, Eskew, Weinert 2025+)

1. Ψ *is indivisible.*
2. *Finite antichains in* Ψ *have infinite BRD.*
3. *Chains of size 2 in* Ψ *have BRD lower bound* 7.

Theorem (Chodounský, Dobrinen, Weinert 2025+)

All finite chains in Ψ *have finite BRD.*
In particular, chains of size 2 *have BRD at most* 7.

1. First BRD results for a Fraïssé-HP class.
2. We seem to have a lower bound proof showing that the upper bounds are exact.
3. We seem to be getting TRS of coding trees for a Fraïssé-HP class.

Arithmetical Aspects of Big Ramsey Degrees

Computability Theory and Reverse Math Results.

The Challenge: Problems on computability theory and reverse math of Ramsey theory on Rado graph, Milliken's Theorem, etc., listed in

• D., Laflamme, Sauer. *Rainbow Ramsey simple structures*, Discrete Mathematics (2016).

• D. *A List of Problems on the Reverse Mathematics of Ramsey Theory on the Rado Graph and on Infinite, Finitely Branching Trees* (arXiv:1808.10227)

• My talk at the Oaxaca 2019 Workshop on Reverse Mathematics of Combinatorial Principles.

The Response

Anglès d'Auriac, Cholak, Dzhafarov, Monin, Patey. *Milliken's tree theorem and its applications: a computability-theoretic perspective*.
AMS Memoirs (2023)

Theorem (Some results in ACDMP)

1. *Big Ramsey degree theorems for the Rationals and the Rado graph are provable in* ACA_0.
2. *There is a computable coloring of pairs in the rationals all of whose subcopies with the minimal number of colors computes* $\mathbf{0}'$.
3. *On the other hand, the Rado graph big Ramsey degrees has cone avoidance for pairs.*

Computability and Big Ramsey Degrees

Theorem (Gill, lower bound, PhD Thesis 2023)

There is a computable coloring of the vertices in $\mathbf{H}_3$ with no computable homogeneous copy.

$\mathrm{FBRD}(\mathbf{H}_3) =$ "$\mathbf{H}_3$ has finite big Ramsey degrees".

Theorem (Anglès d'Auriac, Liu, Mignoty, Patey, APAL 2023)

$\mathrm{FBRD}(\mathbf{H}_3)$ *is provable in* ACA_0^+.

- (ALMP) showed that Carlson–Simpson Thm. is provable in ACA_0^+ and used Hubička's proof that C–S Thm implies $\mathrm{FBRD}(\mathbf{H}_3)$. C–S Thm. implies Hindman's Thm, which is provable in ACA_0^+ by Blass, Hindman, and Hirst.

Theorem (Cholak, D., McCoy, 2025+)

There is computable eventually increasing function $\ell(\mathbf{A})$ and a coloring $c(\mathbf{A})$ such that if $|\mathbf{A}| \geq 2$ then $\ell(\mathbf{A}) > 0$ and every minimal heterogeneous copy w.r.t. $\mathbf{A}$ and $c(\mathbf{A})$ computes $\mathbf{0}^{\ell(\mathbf{A})}$.

Hence, $\mathrm{FBRD}(\mathbf{H}_3)$ *implies* ACA_0'.

- The proof adapts similar colorings of Jockusch utilizing Dobrinen's coding tree representation of $\mathbf{H}_3$.

Theorem (* Cholak, D., Towsner 2025+)

All forcing proofs on coding trees can be done arithmetically.

Concluding Thoughts

- Big Ramsey degrees are characterized by enumerating the structure and categorizing the smallest and slowest essential changes that happen in a substructure.
- Exact big Ramsey degrees and infinite-dimensional Ramsey theory on homogeneous structures inherently involves trees of 1-types.
- A propitious outcome of this study has been the development of varying new methods and new extensive Ramsey theorems.

Expository and survey articles

Nguyen Van Thé. *Structural Ramsey theory with the KPT correspondence in mind*, Habil., Université d'Aix-Marseille, (2013).

Nguyen Van Thé. *A survey on structural Ramsey theory and topological dynamics with the KPT correspondence in mind*, Selected Topics in Combinatorial Analysis, Zb. Rad., (2015).

Dobrinen. *Ramsey theory on infinite structures and the method of strong coding trees*, in Contemporary Logic and Computing, (2020).

Dobrinen. *Ramsey theory of homogeneous structures: current trends and open problems*, Proceedings of the 2022 International Congress of Mathematicians, (2023).

Thank You!